Some items from the lectures on Colley, Section 4.1

TAYLOR POLYNOMIAL APPROXIMATIONS. Students who have taken a course on infinite series have already seen Taylor's formula for functions of one variable, but since such a course is not a prerequisite for this one we shall summarize the main point.

The Mean Value Theorem for single variable functions states that, under suitable conditions, we have

$$f(a+h) \quad = \quad f(a) \ + \ f'(Y)\cdot h$$

where Y is between a and $a+h$. There are several versions of Taylor's Formula for approximating functions by polynomials of degree n, and the following one is an analog of the Mean Value Theorem:

Taylor's Formula. *If f has continuous derivatives of order $\leq n+1$ on some open interval $a-r < a < a+r$, then for all h such that $a+h$ lies in this interval we have*

$$f(a+h) \quad = \quad \sum_{k=0}^{n} \frac{f^{(k)}(a)\cdot h^k}{k!} \ + \ \frac{f^{(n+1)}(Y)\cdot h^{n+1}}{(n+1)!}$$

where $f^{(k)}$ denotes the $k^{\rm th}$ derivative of f with $f^{(0)} = f$, σ denotes the sum of all terms from $k=0$ to n, and Y lies between a and $a+h$.

The summation part of the right hand side is called the $n^{\rm th}$ *degree Taylor polynomial* for f at a, which will sometimes be written $T_n f(x;a)$ or $T_n f(x)$ if a is understood, and the single summand at the end is called the *remainder term* and written $R_n f(x;a)$.

The derivation of this formula can be done in several ways (for example, in Widder's book cited in the text bibliography it is done using integration by parts), but we shall simply assume this formula as known.

Example. Suppose we take $f(x) = \sin x$ with $a = 0$. The the third degree Taylor polynomial approximation is given by $x - \frac{1}{6}x^3$. In order to determine the accuracy of this approximiation, we shall use the full version of the formula with the remainder term

$$\sin x \quad = \quad x \ - \ \frac{x^3}{6} \ + \ \frac{Y^5}{120}$$

where Y is between 0 and x. For the sake of definiteness, we shall take $x = \pi/4$; in order to estimate the accuracy of the third degree approximation, we need to get an upper estimate on the size of the fifth degree term. Since

$$\frac{\pi}{4} \ < \ \frac{4}{5}$$

it follows that Y is between 0 and $\frac{4}{5}$, so a crude upper estimate for the fifth degree term can be obtained by replacing Y with $\frac{4}{5}$. If we do this we see that the maximum error in the approximation is no greater than

$$\frac{4^5}{5^5\cdot 120} \quad = \quad 0.00273066 \ ...$$

and this is well within 1 per cent of the true value, which is $\frac{1}{2}\sqrt{2} \approx 0.707....$

EXTENSION TO FUNCTIONS OF SEVERAL VARIABLES. In this course we shall only need the first and second degree Taylor polynomial approximations; a discussion of higher order Taylor polynomials appears on pages 241–243 of the course text. Also, since we can get by without discussing the form of the remainder term explicitly, we shall pass on trying to describe it here. Further details are also available in Section V.4 (pp. 84–90) of the following online lecture notes for a second linear algebra course:

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

There is a clickable version of this link in the file `weblinks4.pdf` in the course directory.

The first degree Taylor polynomial approximation to a well-behaved function $f(x_1, x_2, \cdots)$ is given by the previous linear approximation formula:

$$T_1 f(\mathbf{a}+\mathbf{h}) \;=\; f(\mathbf{a}) \;+\; \nabla f(\mathbf{a})\cdot\mathbf{h} \;=\; f(\mathbf{a}) \;+\; \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\mathbf{a})\right)\cdot h_i$$

The second degree Taylor polynomial is the sum of this with an appropriate second degree term; here is the explicit formula:

$$T_2 f(\mathbf{a}+\mathbf{h}) \;=\; f(\mathbf{a}) \;+\; \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\mathbf{a})\right)\cdot h_i \;+\; \frac{1}{2}\cdot\left(\sum_{i,j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right)\cdot h_i h_j\right)$$

If $n = 2$ and the variables are given by x and y and $\mathbf{h} = (h_1, h_2)$, then we may rewrite the second degree part in the form

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\,h_1^2 \;+\; 2\cdot\frac{\partial^2 f}{\partial x \partial y}\,h_1 h_2 \;+\; \frac{\partial^2 f}{\partial y^2}\,h_2^2\right)$$

and in three variables x, y, z one can rewrite this similarly using the convention that f_{uv} is the second partial derivative with respect to u and v:

$$\frac{1}{2}\big(f_{xx}h_1^2 + 2f_{xy}h_1h_2 + fyyh_2^2 + 2f_{xz}h_1h_3 + 2f_{yz}h_2h_3 + f_{zz}h_3^2\big)$$

In both of these displayed formulas, the second partial derivatives are evaluated at the reference point $\mathbf{a}$.

Answers to selected exercises from Colley, Section 4.1

2. In this problem, $a = 0$ and $k = 3$, and we have the following:

$$f(x) \;=\; \log_e(1+x) \;\Longrightarrow\; f(0) \;=\; 0$$

$$f'(x) \;=\; \frac{1}{1+x} \;\Longrightarrow\; f'(0) \;=\; 1$$

$$f''(x) \;=\; \frac{-1}{(1+x)^2} \;\Longrightarrow\; f''(0) \;=\; \frac{-1}{1} \;=\; -1$$

$$f'''(x) \;=\; \frac{2}{(1+x)^3} \;\Longrightarrow\; f'''(0) \;=\; \frac{(-1)^2\cdot 2}{1} \;=\; 2$$

Substituting these into the Taylor polynomial formula, we find that the third degree approximation is given by

$$T_3 f(x) \;=\; 0 \;+\; x \;-\; \frac{1}{2}\,x^2 \;+\; \frac{2}{3!}\,x^3 \;=\; x \;-\; \frac{x^2}{2} + \frac{x^3}{3}\,.$$

4. In this problem, $a = 1$ and $k = 3$, and we have the following:

$$\begin{array}{rcccrcl} f(x) & = & \sqrt{x} & \Longrightarrow & f(1) & = & 1 \\ f'(x) & = & \frac{1}{2}\,x^{-(1/2)} & \Longrightarrow & f'(1) & = & \frac{1}{2} \\ f''(x) & = & \frac{-1}{4}\,x^{-(3/2)} & \Longrightarrow & f''(1) & = & \frac{-1}{4} \\ f'''(x) & = & \frac{3}{8}\,x^{-(5/2)} & \Longrightarrow & f'''(1) & = & \frac{3}{8} \end{array}$$

Substituting these into the Taylor polynomial formula, we find that the third degree approximation is given by

$$T_3 f(x) \;=\; 1 \;+\; \frac{1}{2}\,(x-1) \;-\; \frac{1}{8}\,(x-1)^2 \;+\; \frac{1}{16}\,(x-1)^3 \;.$$

8. We shall begin by computing all the first and second partial derivatives of f at $(0,0)$:

$$\begin{array}{rcccrcl} f(x,y) & = & \dfrac{1}{1+x^2+y^2} & \Longrightarrow & f(0,0) & = & 1 \\ f_x(x,y) & = & \dfrac{-2x}{(1+x^2+y^2)^2} & \Longrightarrow & f_x(0,0) & = & 0 \\ f_y(x,y) & = & \dfrac{-2y}{(1+x^2+y^2)^2} & \Longrightarrow & f_y(0,0) & = & 0 \\ f_{xx}(x,y) & = & \dfrac{6x^2-2y^2-2}{(1+x^2+y^2)^3} & \Longrightarrow & f_{xx}(0,0) & = & -2 \\ f_{yy}(x,y) & = & \dfrac{6y^2-2x^2-2}{(1+x^2+y^2)^3} & \Longrightarrow & f_{yy}(0,0) & = & -2 \\ f_{xy}(x,y) & = & \dfrac{8xy}{(1+x^2+y^2)^3} & \Longrightarrow & f_{xy}(0,0) & = & 0 \end{array}$$

Substituting these into the Taylor polynomial formula, we find that the second degree approximation is given by

$$T_2 f(x) \;=\; 1 \;+\; \frac{1}{2}\left(f_{xx}(0,0)x^2 \;+\; 2f_{xy}(0,0)xy \;+\; f_{yy}(0,0)y^2\right) \;=\; 1-(x^2+y^2)\;.$$

This corresponds to the answer one would get by substituting x^2+y^2 into the geometric series for $1/(1-z)$.

10. Once again we shall begin by computing all the first and second partial derivatives of f at $(0,0)$:

$$\begin{array}{rcccrcl} f(x,y) & = & e^{2x+y} & \Longrightarrow & f(0,0) & = & 1 \\ f_x(x,y) & = & 2e^{2x+y} & \Longrightarrow & f_x(0,0) & = & 0 \\ f_y(x,y) & = & e^{2x+y} & \Longrightarrow & f_y(0,0) & = & 0 \\ f_{xx}(x,y) & = & 4e^{2x+y} & \Longrightarrow & e^{2x+y} & = & -2 \end{array}$$

$$\begin{aligned}
f_{yy}(x,y) &= e^{2x+y} \implies f_{yy}(0,0) = -2 \\
f_{xy}(x,y) &= 2e^{2x+y} \implies f_{xy}(0,0) = 0
\end{aligned}$$

Substituting these into the Taylor polynomial formula, we find that the second degree approximation is given by

$$T_2 f(x) = 1 + (f_x(0,0)x + f_y(0,0)y) + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) =$$

$$1 + 2x + y + \frac{1}{2}\left(4x^2 + 2(2)xy + y^2\right) =$$

$$1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} .$$

12. Once again we shall begin by computing the first and second partial derivatives of f at $(0,0,0)$:

$$\begin{aligned}
f(x,y,z) &= \frac{1}{1+x^2+y^2+z^2} \implies f(0,0,0) = 1 \\
f_x(x,y,z) &= \frac{-2x}{(1+x^2+y^2+z^2)^2} \implies f_x(0,0,0) = 0 \\
f_y(x,y,z) &= \frac{-2y}{(1+x^2+y^2+z^2)^2} \implies f_x(0,0,0) = 0 \\
f_z(x,y,z) &= \frac{-2z}{(1+x^2+y^2+z^2)^2} \implies f_x(0,0,0) = 0 \\
f_{xx}(x,y,z) &= \frac{6x^2-2y^2-2z^2-2}{(1+x^2+y^2+z^2)^3} \implies f_{xx}(0,0,0) = -2 \\
f_{yy}(x,y,z) &= \frac{6y^2-2x^2-2z^2-2}{(1+x^2+y^2+z^2)^3} \implies f_{yy}(0,0,0) = -2 \\
f_{zz}(x,y,z) &= \frac{6z^2-2x^2-2y^2-2}{(1+x^2+y^2+z^2)^3} \implies f_{zz}(0,0,0) = -2 \\
f_{xy}(x,y) &= \frac{8xy}{(1+x^2+y^2+z^2)^3} \implies f_{xy}(0,0,0) = 0 \\
f_{yz}(x,y) &= \frac{8yz}{(1+x^2+y^2+z^2)^3} \implies f_{xy}(0,0,0) = 0 \\
f_{xz}(x,y) &= \frac{8xz}{(1+x^2+y^2+z^2)^3} \implies f_{xy}(0,0,0) = 0
\end{aligned}$$

It follows that

$$T_2 f = 1 - \frac{1}{2}(2x^2+2y^2+2z^2) \;\; 1-(x^2+y^2+z^2)$$

and the remarks at the end of the solution to Exercise 8 also apply here.

13. As in many of the preceding exercises, we shall begin by computing first and second partial derivatives of f at $(0,0,0)$:

$$f(x,y,z) = \sin xyz \implies f(0,0,0) = 0$$

$$
\begin{array}{rclcrcl}
f_x(x,y,z) & = & yz\cos xyz & \Longrightarrow & f(0,0,0) & = & 0 \\
f_{xx}(x,y,z) & = & -y^2z^2\sin xyz & \Longrightarrow & f_{xx}(0,0,0) & = & 0 \\
f_{xy}(x,y) & = & z\cos xyz \;-\; xyz^2\sin xyz & \Longrightarrow & f_{xy}(0,0,0) & = & 0
\end{array}
$$

We can now use the fact the $f(x,y,z)$ is symmetric in x, y and z to find the remaining two first partial derivatives of f and the remaining four second partial derivatives of f; these are obtained by interchanging the roles of the three variables consistenly in each of the given expressions, and it follows that all of the first and second partial derivatives at $(0,0,0)$ must be zero. This means that the second degree Taylor polynomial approximation to f is the zero polynomial.

14. Using the computations from Exercise 8 we can write down the Hessian directly:

$$
Hf(0,0) \quad = \quad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} .
$$

15. The first step is to calculate the first and second partial derivatives of f:

$$
\begin{aligned}
f_x(x,y,z) &= 3x^2 + 2xy \\
f_y(x,y,z) &= x^2 - z^2 \\
f_z(x,y,z) &= -2yz + 6z^2 \\
f_{xx}(x,y,z) &= 6x + 2y \\
f_{yy}(x,y,z) &= 0 \\
f_{zz}(x,y,z) &= -2y + 12z \\
f_{xy}(x,y) &= 2x \\
f_{yz}(x,y) &= -2z \\
f_{xz}(x,y) &= 0
\end{aligned}
$$

It follows that the Hessian matrix is given by

$$
\begin{bmatrix} 6x + 2y & 2x & 0 \\ 2x & 0 & -2z \\ 0 & -2z & -2y + 12z \end{bmatrix}
$$

and if we evaluate this at $(1,0,1)$ we find that

$$
Hf(1,0,1) \quad = \quad \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix} .
$$

18. We shall simply write this out using the computations from Exercise 15:

$$
T_2 f \quad = \quad 3 + 3(x-1) + 6(z-1) + \frac{1}{2}\left(6(x-1)^2 + 4(x-1)y - 4y(z-1) + 12(z-1)^2\right)
$$

26. Students are not responsible for knowing how to work this problem.

28. Ditto.

30. Ditto.

31. Ditto.