Some items from the lectures on Colley, Section 4.2

TESTS FOR EXTREMA. In this section we work extensively with second order Taylor polynomial approximations.

For functions of a single variable, the first tests for the maxima and minima of a continuous function on a closed interval are fairly straightforward. There are three types of points one must consider.

(1) Critical points where the derivative is zero.
(2) Critical points where the derivative is undefined.
(3) Boundary points.

An example of the first type is given by $f(x) = x^2$ on the interval $-1 \leq x \leq 1$. In this case the solution to the equation $f'(x) = 0$ is $x = 0$ and in fact x takes its minimum value there. An example of the second type is given by $f(x) = x^{2/3}$. In this case the minimum occurs at $x = 0$ and $f'(0)$ is not defined (the tangent line to the graph is vertical). Finally, in both of the preceding examples the maximum value occurs at the end points.

Sometimes one has functions for which there are *relative* maxima and minima which are distinct from absolute maxima and minima. For example, consider the function $f(x) = x^3 - x$ over the interval $-2 \leq x \leq 2$. The maxima and minima occur at the endpoints, but the equation $0 = f'(x) = 3x^2 - 1$ has the two solutions $x = \pm\sqrt{1/3}$. If we graph the function we see that f has a strict relative maximum at the negative point and a strict relative minimum at the positive point. This means that the values of f near these points are always less than $f\left(-\sqrt{1/3}\right)$ and always greater than $f\left(\sqrt{1/3}\right)$ respectively. for example, this holds if we take all points within $\frac{1}{8}$ of $x = \pm\sqrt{1/3}$.

In fact, there is a **second derivative test** which can often be used to show that there is a relative maximum or minimum. If $f'(x) = 0$ and $f''(x)$ is **negative**, then the function has a relative **maximum** at x, while if $f'(x) = 0$ and $f''(x)$ is **positive**, then the function has a relative **minimum** at x. This can be a little confusing, but one way to remember this is to think of the function x^2, which has a minimum at 0 and whose second derivative is equal to 2 (both at 0 and also everywhere else). If $f''(x) = 0$, then the second derivative test fails and one must look further; for functions of one variable there are simple extensions of the second derivative test which need not concern us here.

The situation is similar for continuous functions of several variables over closed bounded sets of the form $f_i(\mathbf{x}) \geq 0$, where $i = 1, 2, \cdots$, and the inequalities imply that there is some upper bound M for the coordinates of all points in the set (equivalently, there s some upper bound on the distances of all points in the set to some fixed point such as the origin). In this course we shall only consider situations in which the partial derivatives are defined and continuous away from the boundaries.

For functions of several variables, the analog of the equation $f' = 0$ is $\nabla f = \mathbf{0}$, or equivalently the vanishing of all the first partial derivatives at a point.

The analogous second derivative test in several variables is somewhat more complicated. Not surprisingly, it involves all of the second partial derivatives; however, it also involves a relatively substantial amount of linear algebra that is not covered until the end of Mathematics 132; the details appear in the previously cited online document (see the file `answers41.pdf` or `weblinks4.pdf`). Here we shall concentrate on trying to motivate the problems which arise, stating the main results, and working some examples.

Given a function of several variables and a critical point at which $\nabla f(\mathbf{x}) = \mathbf{0}$, the determinant of the Hessian of f at $\mathbf{x}$ plays a role somewhat similar to that of the second derivative in the single variable case. However, the following examples show that the situation is more complicated.

(1) If $f(x,y) = x^2 + y^2$, then the determinant of the Hessian is always equal to 4, and likewise if $f(x,y) = 4 - x^2 - y^2$. The first function has a relative (in fact, absolute) minimum at $\mathbf{0}$ (the only point at which the gradient is $\mathbf{0}$), and similarly the second has a relative (in fact, absolute) maximum at $\mathbf{0}$. Therefore the sign of the determinant of the Hessian does not distinguish between maxima and minima.

(2) If $f(x,y) = x^2 - y^2$, then the determinant of the Hessian is always equal to -4, but the function has neither a relative maximum or minimum at $\mathbf{0}$, which is again the only point at which the gradient is $\mathbf{0}$. More precisely, for all $w \neq 0$ we know that $f(w,0) > 0 = f(0,0)$ and $f(0,w) < 0 = f(0,0)$. Therefore there are situations where the determinant of the Hessian is nonzero but the function has neither a maximum of minimum. If one graphs the surface $z = x^2 - y^2$ along the lines of Figure 2.23 on page 90 of the text, one can see that the graph looks like the surface of a saddle near $\mathbf{0}$, and for this reason one often says that the critical point $\mathbf{0}$ is a **saddle point** for f. This is related to the notion of unstable equilibrium in physics. If we place a marble at $\mathbf{0}$ and push it in the x-direction, it will move upward, but if we push it in the y-direction, it will move downward.

SECOND DERIVATIVE TESTS IN TWO VARIABLES. Suppose that we are given a function $f(x,y)$ defined on an open region U, and $\mathbf{a}$ is a point of U for which $\nabla f(\mathbf{a}) = \mathbf{0}$ (in other words, all the first partial derivatives vanish at $\mathbf{a}$).

> The function f has a **strict relative minimum** at $\mathbf{a}$ provided the second partial derivative $f_{xx}(\mathbf{a})$ is *positive* and the determinant of the Hessian of f at $\mathbf{a}$ is also positive.
>
> The function f has a **strict relative maximum** at $\mathbf{a}$ provided the second partial derivative $f_{xx}(\mathbf{a})$ is *negative* and the determinant of the Hessian of f at $\mathbf{a}$ is positive.
>
> The function f has a **saddle point** at $\mathbf{a}$ provided the determinant of the Hessian of f at $\mathbf{a}$ is negative.
>
> No conclusion can be obtained if the determinant of the Hessian of f at $\mathbf{a}$ is zero (in which case we say there is a **degenerate** critical point).

These conditions exhaust all the possibilities, for if we have a 2×2 matrix in which the upper left entry is zero, then its determinant cannot be positive (why not?).

Example. Describe the critical points of $f(x,y) = x^3 - 3xy + y^3$. — We start by setting $\nabla f = \mathbf{0}$. Since $f_x = 3x^2 - 3y = 0$ and $f_y = -3x + y^2 = 0$, we obtain the equations $x^2 = y$ and $y^2 = x$, so that $x^4 = x$. The latter implies that $x = 0$ or 1, and if we substitute these back into $y = x^2$ we see that the critical points of f are $(0,0)$ and $(1,1)$.

Next, we have to find the Hessian, which means we must first find the second partial derivatives. These are given by $f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = -3$. Therefore the Hessians at the two critical points are given as follows:

$$Hf(0,0) \quad = \quad \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}, \qquad Hf(0,0) \quad = \quad \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

. The determinant of the first Hessian is -9, and hence f has a saddle point at $(0,0)$, while the determinant of the second Hessian is 27. Since $f_{xx}(1,1) = 6 > 0$, it follows that f has a relative minimum at $(1,1)$.

Another example. We shall now work a maximum–minimum problem for a continuous function on a closed bounded set of the type described before.

Find the maximum and minimum values of $f(x,y) = x^2 + y^2 - x - y + 1$ on the closed bounded set defined by $x^2 + y^2 \leq 1$.

There are two parts to this. First, one must find candidates for maxima and minima given by critical points. Then one must examine the boundary to find candidates for critical points on this set.

To find the critical points, we must solve the equation $\mathbf{0} = \nabla f = (2x-1, 2y-1)$. The unique solution to this system of equations is $x = y = \frac{1}{2}$, and we have $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. Next, we must look at the boundary. The way to do this is to represent it by a parametrized curve, for this will translate the boundary problem to an extrema problem for functions of one variable. For instance, we can take the obvious parametrization $\mathbf{x}(t) = (\cos t, \sin t)$. With this parametrization the function on the boundary becomes $g(t) = 2 - \sin t - \cos t$. The maxima and minima for this function are given by setting $g'(t) = 0$, and if we do this we find that the critical points between 0 and 2π are given by $\frac{1}{4}\pi$ and $\frac{5}{4}\pi$. Now the value of g at these points are $2 - \sqrt{2}$ and $2 + \sqrt{2}$ respectively. This tells us that there are three possible values for the maximum and minimum value at f, and to find the absolute maxima and minima we need only put these values in order. However, it is not difficult to see that

$$\frac{1}{2} \quad = \quad f\left(\frac{1}{2}\right) \quad < \quad 2-\sqrt{2} \quad = \quad f\left(\frac{\pi}{4}\right) \quad < \quad 2+\sqrt{2} \quad = \quad f\left(\frac{5\pi}{4}\right)$$

and we can read off the maximum and minimum from this display.

SECOND DERIVATIVE TESTS IN THREE VARIABLES. Suppose now that we are given a function $f(x,y,z)$ defined on an open region U, and $\mathbf{a}$ is a point of U for which $\nabla f(\mathbf{a}) = \mathbf{0}$ (in other words, all the first partial derivatives vanish at $\mathbf{a}$). It will be convenient to introduce some terminology; namely, the *second principal minor* of the Hessian will be the following 2×2 determinant:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

> The function f has a **strict relative minimum** at $\mathbf{a}$ provided the second partial derivative $f_{xx}(\mathbf{a})$ is *positive*, the second principal minor of the Hessian of f at $\mathbf{a}$ is *positive*, and and the determinant of the Hessian of f at $\mathbf{a}$ is also *positive*.
>
> The function f has a **strict relative maximum** at $\mathbf{a}$ provided the second partial derivative $f_{xx}(\mathbf{a})$ is *negative*, the second principal minor of the Hessian of f at $\mathbf{a}$ is *positive*, and and the determinant of the Hessian of f at $\mathbf{a}$ is *negative*.
>
> The function f has a **saddle point** at $\mathbf{a}$ provided the determinant of the Hessian of f at $\mathbf{a}$ is nonzero and neither of the preceding statements hold..
>
> No conclusion can be obtained if the determinant of the Hessian of f at $\mathbf{a}$ is zero (in which case we say there is a **degenerate** critical point).

Once again, the proof requires some input from linear algebra beyond the scope of this course. The previously cited online reference gives a full discussion of this test and its extensions to functions of four or more variables.

Answers to selected exercises from Colley, Section 4.2

2. The gradient of g is given by $(g_x, g_y) = (2x+2, -4y)$, so that $g_{xx} = 2$, $g_{xy} = 0$, and $g_{yy} = -4$.

(a) To find the critical point, set $\nabla g = \mathbf{0}$, obtaining the equations $2x + 2 = 0 = 4y$, which imply that there is one critical point given by $(-1, 0)$.

(b) Direct computation shows that $\Delta g = g(-1+\Delta x, \Delta y) - g(-1, 0)$ is equal to $(\Delta x)^2 - 2(\Delta y)^2$. Therefore small changes in x alone will result in an increase in the value of g and small changes in y will result in a decrease in the value of g. Therefore f must have a saddle point at $(-1, 0)$.

(c) The Hessian is given by

$$Hg(-1,0) \quad = \quad \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

and since the determinant of the Hessian is -8 it follows from the second derivative test that g has a saddle point at $(-1, 0)$.

4. We have

$$\nabla f \quad = \quad \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1} \right)$$

and hence the only critical point of f is at the origin. The second derivatives are

$$f_{xx} \quad = \quad \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}, \qquad f_{yy} \quad = \quad \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}, \qquad f_{xy} \quad = \quad \frac{4xy}{(x^2+y^2+1)^2}.$$

At the origin, the Hessian is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and since the determinant of the Hessian and the upper left entry are both positive, it follows that f has a relative minimum at $(0, 0)$.

5. The gradient of f is $(2x - 6y + 3, 3y^2 - 6x + 6)$, so the critical points satisfy the equations $2x = 6y - 3$ and $0 = 3y^2 - 6x + 6$, or equivalently $0 = y^2 - 2x + 2$. Substituting, we obtain the equation $0 = y^2 - 6x + 5 = (y-1)(y-5)$. Therefore we have critical points at $(3/2, 1)$ and $(27/2, 5)$. The second derivatives are $f_{xx} = 2$, $f_{yy} = 6y$, and $f_{xy} = -6$. Thus the upper left entry is always positive, and the determinant of the Hessian, which is $12y - 36$, will be positive when $y = 5$ and negative when $y = 1$; by the second derivative test, f has a saddle point at $(3/2, 1)$ and a local minimum at $(27/2, 5)$.

10. The gradient of f is given by $(1 - 2xy - y^2, 1 - 2xy - x^2)$. The critical points for f satisfy the equations $1 - 2xy - y^2 = 0 = 1 - 2xy - x^2$, and if we subtract $1 - 2xy$ from both sides we obtain $x^2 = y^2$, so that $x = \pm y$. If $x = y$, then $0 = 1 - 3x^2$, so that $x = y = \pm 1/\sqrt{3}$. If $x = -y$, then $0 = 1 + x^2$, which has no real solutions. So the critical points for f are $\pm(1/\sqrt{3}, 1/\sqrt{3})$. The second order partial derivatives are $f_{xx} = -2y$, $f_{yy} = 2x$, and $f_{xy} = -2x - 2y$. The upper left entry of the Hessian is equal to $-2y$, and the determinant of the Hessian is equal to $-4x^2 - 4xy - 4y^2$. At the critical points the latter is negative and therefore f has a saddle point at both critical points $\pm(1/\sqrt{3}, 1/\sqrt{3})$.

11. The gradient of f is given by $(2x - 2xy, -3y^2 - x^2 + 1$. The first coordinate may be rewritten as $2x(1-y)$, and this is zero only if $x = 0$ or $y = 1$. When $x = 0$, y must be $\pm 1/\sqrt{3}$. No solution corresponds to $y = 1$, so the critical points for f are $(0, \pm 1/\sqrt{3})$. The second derivatives are $f_{xx} = 2 - 2y$, $f_{yy} = -6y$, and $f_{xy} = -2x$. Thus the upper left entry of the Hessian is $2 - 2y$ and the determinant of the Hessian is $-12y + 12y^2 - 4x^2)$. At $(0, -1/\sqrt{3})$ the upper left entry and determinant are positive, and therefore f has a local minimum at $(0, -1/\sqrt{3})$. At $(0, 1/\sqrt{3})$, the upper left entry of the Hessian is positive and the determinant is negative, so that f has a saddle point at $(0, 1/\sqrt{3})$.

12. The gradient of f is $\left((e^{-x}(2x - x^2 - 3y^2), 6ye^{-x}\right)$; since the second coordinate vanishes if and only if $y = 0$, the latter must hold at a critical point for f. Using this, we see that the first coordinate is either 0 or 2, so that there are critical points at $(0,0)$ and $(2,0)$. The second derivatives are $f_{xx} = (2 - 4x + x^2 + 3y^2)e^{-x}$, $f_{yy} = -6e^{-x}$, and $f_{xy} = -6ye^{-x}$. At $(0,0)$, the upper left entry and determinant of the Hessian are positive so f has a local minimum there. At $(2,0)$, the upperleft entry and determinant of the Hessian are negative so f has a saddle point there.

16. The gradient of f is $(2x \cos z, 4y \cos z, -(x^2 + 2y^2 + 1) \sin z)$. The third coordinate is zero if and only if $z = n\pi$, and since $\cos n\pi \neq 0$ it follows that if $\nabla f = \mathbf{0}$ then $x = y = 0$. Therefore the critical points are given by $(0, 0, n\pi)$. The second derivatives are $f_{xx} = 2 \cos z$, $f_{yy} = 4 \cos z$, $f_{zz} = -(x^2 + 2y^2 + 1) \cos z$, $f_{xy} = 0$, $f_{xz} = -2x \sin z$ and $f_{yz} = -4y \sin z$. This means that the second principal minor given by $8 \cos 2z$, and the determinant of the Hessian at the critical point $(0, 0, n\pi)$ is $(-1)^{n+1}8$. Similarly, the second principal minor is always 8 and the upper left term is $(-1)^n 2$. Hence the signs of the upper left entry and the determinant are opposite, which means that these critical points are nondegenerate but cannot be relative maxima or minima. The only remaining possibility is that they are all saddle points.

19. The gradient of f is $(y + z - x^{-2}, x + 2z, x + 2y)$. If $\nabla f = \mathbf{0}$, then $z + 2z = 0 = x + 2y$ implies that $y = z$, which in turn implies that $2z = -x$ and $2z = x^{-2}$, so that $-x = ^{-2}$, which means that $x = -1$. Therefore f has a critical point at $(-1, \frac{1}{2}, \frac{1}{2})$ and nowhere else. The Hessian is given by

$$Hf(x, y, z) \quad = \quad \begin{bmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

so that the upper left entry at the critical point is -2, the second principal minor is -1, and the determinant is 12. It follows that the critical point is degenerate but the criteria for a relative maximum or minimum are not met (the second principal minor is negative). Therefore the critical point is a saddle point.

22. (a) The gradient of f is $(2kx - y, -2x + 2ky)$, so that the origin is a critical point for every value of k. The Hessian is

$$\begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix}$$

so the upper left entry is $2k$ and the determinant is $4k^2 - 4$. In oder for f to have a non-degenerate local maximum or minimum the determinant must be positive, which is equivalent to $k^2 - 1 > 0$, so that either $k > 1$ or $k < -1$. If $k > 1$, then the upper left entry is positive and the origin is a non-degenerate local minimum. If $k < -1$, then the upper left entry is negative and the origin is a non-degenerate local maximum.

(b) The gradient of f is $(2kx+kz, -2z-2y, kx-2y+kz)$, and the Hessian is

$$\begin{bmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{bmatrix}.$$

First note that the upper left entry is $2k$ and the second principal minor is $-4k$. These have opposite signs so a nondegenerate local minimum is not possible. In order to have a nondegenerate local maximum the upper left entry must be negative and the second principal minor must be positive, so that $k > 0$. The determinant of the Hessian must also satisfy $2k(-k-4) < 0$, so that k ¡ -4. Therefore we have a nondegenerate local maximum when $k < -4$.

29. We shall actually minimize the square of the distance (*i.e.*, the sum of the squares of the differences in each direction). Since $D(x, y) = x^2 + y^2 + (3x - 4y - 24)^2$, its gradient is $(20x - 24y - 144, 34y - 24x + 192)$. If we set these equal to 0 and solve, we find that the point on the plane closest to the origin is $(36/13, -48/13, -12/13)$.

32. The fly does not need to walk around the metal plate when searching for the hottest and coldest points. The temperature is $T(x, y) = 2x^2 + y^2 - y - 3$ so its gradient is $(4x, 2y - 1)$. There is a critical point for T at $(0, 1/2)$ and $T(0, 1/2) = 2.75$. To check the temperature of the boundary we note that it is a unit disk and so $x = \cos\theta$ and $y = \sin\theta$. We can rewrite

$$T(\theta) \quad = \quad 2\cos^2\theta \ + \ \sin^2\theta \ - \ \sin\theta \ + \ 3 \quad = \quad \cos^2\theta \ - \ \sin\theta \ + \ 4$$

and accordingly we also have

$$T'(\theta) \quad = \quad -2\cos\theta\sin\theta \ - \ \cos\theta \quad = \quad -\cos\theta_(\sin\theta + 1)\ .$$

Therefore, there are critical points on the boundary when $\cos\theta = 0$ (so $\theta = \pi/2$ or $3\pi/2$) and when $\sin\theta = -1/2$ (so $k = 7\pi/6$ or $11\pi/6$). Checking the values we see that $T(\pi/2) = 3$, $T(3\pi/2) = 5$ and $T(7\pi/6) = T(11\pi/6) = 21/4$. We conclude that the coldest spot on the plate is at $(0, 1/2)$ where the temperature is $11/4$ and the two hottest spots are at $(\pm\sqrt{3}/2, -1/2)$, where the temperature is $21/4$.

33. Since the function $f(x, y)$ is the product $g(x)\,h(y)$ of a function of where the values of g and h range between -1 and 1, we can analyze this example without results from multivariable calculus. The maximum value for f is 1 and the minimum value for f is -1. The absolute maximum occurs at $(\pi/2, 0)$, $(\pi/2, 2\pi)$, and $(3\pi/2, \pi)$. The absolute minimum is achieved at $(3\pi/2, 0)$, $(3\pi/2, 2\pi)$, and $(\pi/2, \pi)$.

34. The gradient of f is $(-2\sin x, 3\cos y)$, so the "ordinary" critical points on the solid rectangle defined by $0 \le x \le 4$, $0 \le y \le 3$ are at $(0, \pi/2)$ and $(\pi, \pi/2)$ (in fact, the second one is the only critical point which is actually in the interior of the rectangle.) Next we look at the boundary of the rectangle. It is convenient to break this up into four parametrized curves corresponding to the four sides of the rectangle:

$g_1(x) = f(x, 0) = 2\cos x$, so that $g_1'(x) = -2\sin x$ and there are critical points at $(0, 0)$ and $(\pi, 0)$.

$g_2(x) = f(x, 3) = 2\cos x + 3\sin 3$, so that $g_2'(x) = -2\sin x$ and there are critical points at $(0, 3), (\pi, 3)$.

$g_3(y) = f(0, y) = 2 + 3\sin y$, so that $g_3'(y) = 3\cos y$ and there is a critical point at $(0, \pi/2)$.

$g_4(y) = f(4, y) = 2\cos 4 + 3\sin y$, so that $g_4'(y) = 3\cos y$ and there is a critical point at $(4, \pi/2)$.

Now we compare values:

$$
\begin{aligned}
f(0, \pi/2) &= 5 \\
f(\pi, \pi/2) &= 1 \\
f((0, 0) &= 2 \\
f(\pi, 0) &= -2 \\
f(0, 3) &= 2 + 3\sin 3 \approx 2.423 \\
f(\pi, 3) &= -2 + 3\sin 3 \approx -1.577 \\
f(4, \pi/2) &= 2\cos 4 + 3 \approx 1.693 \\
f(4, 0) &= 2\cos 4 \approx -1.307 \\
f(4, 3) &= 2\cos 4 + 3\sin 3 \approx -0.884
\end{aligned}
$$

Thus the absolute minimum occurs at $(\pi, 0)$ and is -2. Similarly, the absolute maximum occurs at $(0\pi/2)$ and is 5.

46.(a) To simplify the discussion we shall assume that f' is continuous. If f has a local maximum at x_0 and no other critical points, then f' must be nonzero everywhere else. It follows that either f' is always positive or negative to the left of x_0 and likewise to the right of x_0; if it changed signs to the left or right of x_0, then there would be a point where the derivative would be equal to zero. We claim that the signs of the derivative on opposite sides of x_0 must be different. If they are the same, then there are two cases depending upon whether the common signs are positive or negative.

If f' is positive on both sides of x_0, then by the Mean Value Theorem we know that f is strictly increasing on the rays $x \leq x_0$ and $x \geq x_0$. Therefore if $y < x_0 < z$ we have $f(y) < f(x_0) < f(z)$, and this means that f cannot have a strict relative maximum or minimum at x_0. Likewise, if f' is negative on both sides of x_0 and $y < x_0 < z$ then $f(y) > f(x_0) > f(z)$, so once again there is no relative maximum or minimum at x_0. Therefore the derivative is positive on one side of x_0 and negative on the other.

If $f'(t) < 0$ for $t < x_0$ and $f'(t) > 0$ for $t > x_0$, by the Mean Value Theorem we know that f is strictly decreasing on the ray $x \leq x_0$ and strictly increasing on the ray $x \geq x_0$. Therefore if $y < x_0 < z$ we have $f(x_0) > f(y), f(z)$ so that there is an absolute maximum at x_0. Similarly, if $f'(t) > 0$ for $t < x_0$ and $f'(t) < 0$ for $t > x_0$, by the Mean Value Theorem we know that f is strictly increasing on the ray $x \leq x_0$ and strictly decreasing on the ray $x \geq x_0$. Therefore if $y < x_0 < z$ we have $f(x_0) < f(y), f(z)$ so that there is an absolute minimum at x_0.

(b) The gradient of f is $(3ye^x - 3e^{3x}, 3e^x - 3y^2)$. Solving the equation $\nabla f = \mathbf{0}$, we see that $y = 0$ or $y = 1$. However, we must have $y \neq 0$ because $e^x = y^2$. Thus te only critical point for f is at $(0, 1)$ and $f(0, 1) = 1$. Also, the lower left entry of the Hessian at $(0, 1)$ is -6 and the determinant of the Hessian at $(0, 1)$ is 27, so that f has a local maximum at $(0, 1)$. But the y-axis we have $f(0, y) = 3y - 1 - y^3$, so as $y \to -\infty$ we see that $f(0, y)$ goes to $+\infty$ (in other words, it increases without bound).