Some items from the lectures on Colley, Section 4.3

REGULAR CONSTRAINTS. The central theme of this section is to find maximum and minimum values of a function $f(x, y, z)$ subject to some sort of reasonable constraint of the form $g(x, y, z) = 0$. Specifically, the constraint is reasonable if it satisfies a basic regularity condition: *If* $g(x, y, z) = 0$, *then* $\nabla g(x, y, z) \neq \mathbf{0}$.

MOTIVATING THE LAGRANGE MULTIPLIER RULE. Although the method is extremely useful and very powerful, the reasons behind it are probably not immediately clear, so we shall look at a simple example. Suppose that the constraint is given by an equation of the form $g(x, y, z) = z - h(x, y) = 0$ and we want to maximize the function $f(x, y, z) = z$ on this surface; notice that this is a regular constraint because the z-coordinate of ∇g is always 1. Clearly the given problem is equivalent to looking for maximum and minimum values of $h(x, y)$, and a key step to solving such a problem is to look at the points where the gradient satisfies $\nabla h(x, y) = \mathbf{0}$. The latter in turn is equivalent to the condition that the normal to the surface ∇g is a nonzero multiple of $(0, 0, 1)$, which is equal to ∇f. Since ∇g is the normal to the tangent plane of the constraint set, it follows that ∇f is perpendicular to the tangent plane in this example. In fact this holds more generally:

THEOREM. *Suppose that* $\mathbf{p}$ *satisfies the regular constraint* $g(\mathbf{p}) = 0$ *and gives a maximum or minimum value for a reasonable function* f *over the surface* Σ *of points satisfying the constraint. Then* ∇f *is perpendicular to the tangent plane of* Σ *at* $\mathbf{p}$, *and accordingly* ∇f *is a multiple of* ∇g.

This is illustrated in Figure 4.27 on page 263 of the text.

The theorem implies that there is a scalar λ such that $\nabla f = -\lambda \nabla g$, or equivalently $\nabla(f + \lambda g) = \mathbf{0}$. Combining this with the constraint equation $g(x, y, z) = 0$, we obtain four equations in the four unknowns x, y, z, λ, and in favorable cases (like most textbook problems) we can check that there are finitely many solutions to such a system and solve for them explicitly.

Example. We shall work Exercise 29 from the preceding section using Lagrange multipliers: *Find the point on the plane* $3x - 4y - z = 24$ *which is closest to the origin.*

First of all, we must translate this into a constrained optimization problem; namely, minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $0 = g(x, y, z) = 3x - 4y - z - 24$.

The Lagrange Multiplier Method requires us to solve the system of equations given by $\nabla(f + \lambda g) = \mathbf{0}$ and $g(x, y, z) = 0$. We need to write out the left hand side of the vector equation explicitly:

$$\nabla(f + \lambda g) \quad = \quad \nabla\big((x^2 + y^2 + z^2) \ + \ \lambda(3x - 4y - z - 24)\big) \quad =$$

$$(2x + 3\lambda, 2y - 4\lambda, 2z + \lambda)$$

This vector equation implies that $x = -\frac{3}{2}\lambda$, $y = 2\lambda$, and $z = -\frac{1}{2}\lambda$. We can substitute these values into the fourth equation in order to find λ:

$$0 \quad = \quad g\left(-\frac{3}{2}\lambda, 2\lambda, -\frac{1}{2}\lambda\right) \quad = \quad -\frac{9\lambda}{2} \ - \ 8\lambda \ - \ \frac{\lambda}{2} - 24 \quad = \quad -\frac{26\lambda}{2} \ - \ 24$$

This means that $\lambda = -24/13$, and we can now substitute these values into the equations for x, y, z to see that

$$z \quad = \quad -\frac{12}{13}\,, \qquad y \quad = \quad -\frac{48}{13}\,, \qquad x \quad = \quad \frac{36}{13}\,.$$

One can check this answer geometrically; the shortest distance is realized by a perpendicular from the origin to the plane, and in fact the value for (x, y, z) obtained above is a nonzero multiple of the unit normal vector(s) to the plane.

A word problem. We wish to construct a box whose volume is 288 cubic inches. The material for the bottom costs \$5 per square foot, while the material for the top and sides costs only \$3 per square foot. Find the dimensions that will minimize the cost of materials.

Let x, y, z denote the dimensions of the box, where x, y denote the dimension of the bottom and top. From the nature of the problem we know that x, y, z must all be positive, and the volume constrain means that $xyz = 288$. The area of the bottom and top is xy, while two of the sides have area xz and the remaining two sides have area yz. Therefore the cost of materials for the sides of the box is given by

$$f(x, y, z) \quad = \quad 5xy \; + \; 3xy \; + 6xz + 6yz$$

(the first term is the cost of the bottom, the second is the cost of the top, and the remaining terms are the costs of the two pairs of parallel sides — draw a picture to visualize this!). In this case the vector equation becomes

$$\mathbf{0} \quad = \quad \nabla(f - \lambda g) \quad =$$

$$\nabla\Big((5xy \; + \; 3xy \; + 6xz + 6yz) - \lambda xyz \Big) \quad = \quad (8y + 6z - \lambda yz, 8x + 6z - \lambda xz, 6x + 6y - \lambda xy)$$

This leads to the equations

$$\frac{8}{z} + \frac{6}{y} \quad = \quad -\lambda \; , \quad \frac{8}{z} + \frac{6}{x} \quad = \quad -\lambda \; , \quad \frac{6}{x} + \frac{6}{y} \quad = \quad -\lambda \; .$$

The first two combine to show that $x = y$ (recall that x, y, z are positive so there are no problems with zeros in the denominators). Similarly, the second and third combine to yield

$$\frac{z}{8} \quad = \quad \frac{x}{6} \; , \quad \text{so that} \quad z \quad = \quad \frac{4}{3} x \; .$$

This means that the volume constraint may be rewritten in the form $288 = \frac{4}{3}x^3$, and the unique positive solution to this equation is $x = y = 6$ and $y = 8$.

Finally, we should find the minimum cost which is realized for these dimensions. If we substitute the values for x, y, z from the previous discussion we find that the cost is $f(6, 6, 8) = 864$.

MULTIPLE CONSTRAINTS. Sometimes we are given problems with more than one constraint. For functions of three variables, this happens if we have two constraints $g_1(x, y, z) = g_2(x, y, z) = 0$ which are regular in the sense that $\nabla g_1(x, y, z)$ and $\nabla g_2(x, y, z)$ are not multiples of each other if $g_1(x, y, z) = g_2(x, y, z) = 0$. In this case, if we have a maximum or minimum at $\mathbf{p}$ subject to the constraints then it turns out that $\nabla f(\mathbf{p})$ can be written in the form $\lambda_1 \nabla g_1(\mathbf{p}) + \lambda_2 \nabla g_2(\mathbf{p})$ for suitably chosen scalars λ_1 and λ_2. We then end up with a system of **five** equations in $x, y, z, \lambda_1, \lambda_2$ given by $g_1 = g_2 = 0$ and $\nabla(f + \lambda_1 g_2 + \lambda_2 g_2) = \mathbf{0}$. Once again we solve this system of equations to find the constrained maxima and minima.

There are analogs of the Lagrange Multiplier method for equations in any finite number of variables with multiple constrains satisfying the corresponding regularity conditions, but we shall not attempt to discuss them here.

NOTES. There is a corresponding theory for functions of two variables with one constraint; the only difference is that there is no equation involving partial derivatives with respect to the third variable.

In some cases it is more convenient to replace equations like $\nabla(f + \lambda g) = \mathbf{0}$ with equations of the form $\nabla(f - \lambda g) = \mathbf{0}$. The values of x, y, z obtained as solutions in both cases are the same, but of course the value for λ changes sign.

Answers to selected exercises from Colley, Section 4.3

9.(a) The function is $f(x, y) = x^2 + y$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 1$. The associated equation $\nabla(f - \lambda g) = \mathbf{0}$ now yields the following system of equations in x, y, λ:

$$2x = 2x\lambda \ , \qquad 1 = 4\lambda \ , \qquad x^2 + 2y^2 = 1$$

From the first equation, we know that $2x(1 - \lambda) = 0$, so that either $x = 0$ or $\lambda = 1$. If $\lambda = 1$, then $y = \frac{1}{4}$, so $x = \pm\sqrt{7/8}$. If $x = 0$, then $y = \pm\frac{1}{2}$. Thus the critical points are $(\pm\sqrt{7/8}, \frac{1}{4})$ and $(0, \pm\frac{1}{2})$.

(b) To apply the second derivative test, we need to examine the Hessian of the function $L(\lambda; x, y) = x^2 + y - \lambda(x^2 + 2y^2 - 1)$. The Hessian is given by

$$HL(\lambda; x, y) = \begin{bmatrix} 0 & -2x & -4y \\ -2x & 2 - 2\lambda & 0 \\ -4y & 0 & \lambda \end{bmatrix} .$$

One can now substitute the critical points to find that there are local maxima at $(\pm\sqrt{7/8}, \frac{1}{4})$ and local minima at $(0, \pm\frac{1}{2})$.

17. The symmetry of the problem suggests the answer, but we are maximizing $f(x, y, z) = xyz$ subject to the constraint $0 = g(x, y, z) = x + y + z - 18$. The associated equation $\nabla(f - \lambda g) = \mathbf{0}$ now yields the following system of equations in x, y, z, λ:

$$xy = xz = yz = \lambda \ , \qquad x + y + z = 18$$

Since we want x, y, z to be positive, the solution has the form $x = y = z$, which implies that $3x = 18$ the maximum product occurs at the point $(6, 6, 6)$.

23. We want to minimize $M(x, y, z) = xz - y^2 + 3x + 3$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 9$. This is given by solutions to the system

$$z + 3 = \lambda x, \quad -2y = \lambda y, \quad x = 2\lambda z, \qquad x^2 + y^2 + z^2 = 9 .$$

The second equation implies that either $y = 0$ or $\lambda = -1$. In the first case, we have $z = -3$, and from this we get $(0, 0, -3)$ and $(\pm\sqrt{3}/2, 0, 3/2)$ as critical points. If $\lambda = -1$, we find the critical points are $(-2, 2, 1)$ and $(-2, -2, 1)$. Comparing values of M, the minimum of -9 is attained at either $(-2, 2, 1)$ or $(-2, -2, 1)$.

29. We want to find the extrema for $f(x, y, z) = z$ subject to the constraints $0 = g_1(x, y, z) = x^2 + y^2 - z$ and $0 = g_2(x, y, z) = x + y + 2z = 2$. In this example the vector equation is

$$\mathbf{0} = \nabla(f + \lambda_1 g_1 + \lambda_2 g_2) = (2x\lambda_1 + \lambda_2, 2y\lambda_1 + \lambda_2, 1 + \lambda_1 + 2\lambda_2) .$$

The vanishing of the first two coordinates implies that $2x\lambda_1 = 2y\lambda_1 = -\lambda_2$, and this means that either $\lambda_1 = 0$ or $x = y$. We can exclude the case $\lambda_1 = 0$ because it leads to $-\lambda_2 = 2y \cdot 0$, and this contradicts the third coordinate equation $1 + \lambda_1 + 2\lambda_2 = 0$. Substituting this into the first constraint, we see that $z = 2x^2$. Substituting the equation $x = y$ into the second constraint we obtain $x + z = 1$ and $2x^2 + x - 1 = 0$. The Quadratic Formula now implies that

$$x \quad = \quad \frac{-1 \ \pm \ 3}{4}$$

which in turn yields $x = -1$ or $x = \frac{1}{2}$. By previous steps we also know that $y = -1$ and $y = \frac{1}{2}$ in these respective cases, and also that $z = 2$ and $z = \frac{1}{2}$ respectively. If we evaluate the function at these two critical points, we find that the height takes a maximum value of 2 at $(-1, -1, 2)$ and a minimum value of $\frac{1}{2}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.