Answers to selected exercises from Colley, Section 4.6

2.(a) If $f(\mathbf{x}) = e^{-|\mathbf{x}|^2}$, then the i^{th} partial derivative is $-2x_i e^{-|\mathbf{x}|^2}$, and it is zero if and only if $x_i = 0$. Therefore the only critical point is at the origin.

(b) If $i \neq j$, then $f_{ij} = 4x_i x_j e^{-|\mathbf{x}|^2}$, so $f_{ij}(\mathbf{0}) = 0$. Also, $f_{ii} = (-2 + 4x_i^2)e^{-|\mathbf{x}|^2}$, so so $f_{ii}(\mathbf{0}) = -2$. Thus the Hessian has -2's down the diagonal and zeros elsewhere, and if we compute determinants we see that $\mathbf{0}$ must be a relative maximum (in fact, it is an absolute maximum).

12. Students are not responsible for knowing how to work this problem.

15. The problem is to minimize $M(x, y, z) = x^2 + y^2 + z^2$ subject to $0 = g(x, y, z) = x^2 - (y - z)^2 - 1$. The vector equation from the Lagrange Multiplier Rule is $\mathbf{0} = \nabla(M - \lambda g)$ which translates to the following system of scalar equations:

$$2x = 2\lambda x \;, \quad 2y = -2\lambda(y - z) \;, \quad 2z = 2\lambda(y - z) \;, \qquad x^2 - (y - z)^2 = 1 \;.$$

Since the last equation implies that $x \neq 0$, the first equation gives us that $\lambda = 1$, so that $y = z = 0$ and $x = \pm 1$. Therefore the minimum distance is 1.

20. (a) This is a straightforward exercise in trigonometry. For each section the time is the distance divided by the rate and the hypotenuse is the altitude divided by the cosine of the angle that is formed by the altitude and the hypotenuse. This yields the formula

$$T(\theta_1, \theta_2) = \frac{a}{v_1 \cos\theta_1} + \frac{b}{v_2 \cos\theta_2} \;.$$

(b) We are minimizing time subject to the constraint that the horizontal separation is constant:

$$H(\theta_1, \theta_2) = a \tan\theta_1 + b \tan\theta_2 = c \;.$$

The associated vector equation $\mathbf{0} = \nabla(T - \lambda H)$ yields the following two additional equations:

$$\frac{a \sin\theta_1}{v_1 \cos^2\theta_1} = \frac{\lambda a}{\cos^2\theta_1} \;, \qquad \frac{b \sin\theta_2}{v_2 \cos^2\theta_2} = \frac{\lambda b}{\cos^2\theta_2}$$

Since the angles θ_i are acute, we can cancel the terms $\cos^2\theta_i$ from both of these equations to obtain

$$\frac{a \sin\theta_1}{v_1} = \lambda a \;, \qquad \frac{b \sin\theta_2}{v_2} = \lambda b \;.$$

Dividing the first equation by the second, we obtain

$$\frac{v_2 a \sin\theta_1}{v_1 b \sin\theta_2} = \frac{\lambda a}{\lambda b} = \frac{a}{b}$$

and if we multiply both sides by

$$\frac{v_1 b}{v_2 a}$$

then we obtain Snell's Law as stated in the text.