NAME: ______________________________

Mathematics 10A–001, Fall 2008, Examination 2

Answer Key

Reference material

Spherical to rectangular coordinates.

$$
\begin{aligned}
x &= \rho \cos\theta \sin\phi \\
y &= \rho \sin\theta \sin\phi \\
z &= \rho \cos\phi \\
\rho^2 &= x^2 + y^2 + z^2
\end{aligned}
$$

Selected values of trigonometric functions.

$$
\begin{aligned}
\sin 30^\circ &= \cos 60^\circ = \tfrac{1}{2} \\
\sin 45^\circ &= \cos 45^\circ = \tfrac{\sqrt{2}}{2} \\
\sin 60^\circ &= \cos 30^\circ = \tfrac{\sqrt{3}}{2}
\end{aligned}
$$

Degrees to radians conversion.

$180^\circ = \pi$ radians.

1. [*15 points*] Find spherical coordinates for the point whose rectangular coordinates are $\frac{1}{4}(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$.

SOLUTION

Using the formula for ρ on the preceding sheet we find that

$$\rho \quad = \quad \frac{1}{4} \cdot \sqrt{2 + 1 + (4 \cdot 3)} \quad = \quad \frac{1}{4} \cdot \sqrt{16} \quad = \quad 1 \ .$$

Next we solve for z using the formula on the preceding sheet, obtaining the equation

$$\cos\phi \quad = \quad \frac{z}{\rho} \quad = \quad \frac{\sqrt{3}}{2} \quad = \quad \cos(\pi/6)$$

so that $\phi \quad = \quad \pi/6$. Since $\sin(\pi/6) = \frac{1}{2}$, the equations for θ become

$$\cos\theta \quad = \quad \frac{x}{(1/2)} \quad = \quad 2x \ , \qquad \sin\theta \quad = \quad \frac{y}{(1/2)} \quad = \quad 2y$$

and since $x = \frac{1}{4}\sqrt{2} = y$, this means that $\cos\theta = \frac{1}{2}\sqrt{2} = \sin\theta$, which in turn means that $\theta \quad = \quad \pi/4$.

2. [*10 points*] The polar coordinate equation

$$r = \frac{5}{\sin\theta - 2\cos\theta}$$

defines a line. Find the equation of this line in rectangular coordinates.

SOLUTION

Multiply both sides of the equation by the denominator to clear it of fractions. This yields the equation

$$r\sin\theta - 2r\cos\theta = 5$$

and if we use the conversion formulas for polar and rectangular coordinates we obtain the equation $y - 2x = 5$.

3. [*20 points*] If $u(x,y) = x^3 - 3xy^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

SOLUTION

The first partial derivatives are

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 , \qquad \frac{\partial u}{\partial y} = -6xy$$

so the "pure" second partial derivatives are

$$\frac{\partial^2 u}{\partial x^2} = 6x , \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

and if we add these together we obviously get zero.

4. [*10 points*] If $f(x, y) = e^{x^2+y^2}$, find the mixed partial derivative

$$\frac{\partial^2 f}{\partial x\, \partial y}\ .$$

SOLUTION

If we differentiate f with respect to y we obtain $2y\, e^{x^2+y^2}$, and if we now differentiate the latter with respect to x we obtain $2y\left(2x\, e^{x^2+y^2}\right)$, so the final answer is $4xy \cdot e^{x^2+y^2}$.

5. [*20 points*] Let S denote the surface defined by the equation $x^3 + y^5 + z^7 - 3 = 0$. Find the equation of the tangent plane to S at $(1, 1, 1)$.

SOLUTION

If $f(x, y, z) = x^3 + y^5 + z^7 - 3$, then $\nabla f = (3x^2, 5y^4, 7z^6)$, so that $\nabla f(1, 1, 1) = (3, 5, 7)$ and as such it is nonzero. Therefore the equation of the tangent plane is given by

$$\nabla f(3, 5, 7) \cdot (x, y, z) \quad = \quad \nabla f(3, 5, 7) \cdot (1, 1, 1) \quad =$$

$$3x \ + \ 5y \ + \ 7z \quad = \quad 3 \ + \ 5 + \ 7 \quad = \quad 15 \ .$$

6. [*15 points*] Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in the direction $\mathbf{u} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ at the point $(1, 1, 2)$.

SOLUTION

The formula for the directional derivative is

$$\nabla f(1, 1, 2) \cdot \mathbf{u}$$

so we need to find the gradient first. But $\nabla f = (2x, 2y, 2z)$, so that $\nabla f(1, 1, 2) = (2, 2, 4)$. Therefore the directional derivative is equal to

$$(2, 2, 4) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \quad = \quad \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \quad = \quad 4 \, .$$

7. [*10 points*] Find the length of the piece of the spiral curve with parametric equations $\mathbf{x}(t) = (\cos t, \sin t, t)$ from $t = 0$ to $t = 2\pi$.

SOLUTION

The length is given by

$$L = \int_0^{2\pi} |\mathbf{x}'(t)|\,dt$$

and $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$ means that

$$|\mathbf{x}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

so the length equals

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi\ .$$