Classification of quadric surfaces

Both plane and solid analytic geometry spend considerable time discussing the sets of points in $\mathbf{R}^2$ and $\mathbf{R}^3$ whose coordinates satisfy some quadratic polynomial equation, which can be written in the form

$$\sum_i a_i x_i^2 + \sum_{i<j} 2b_{i,j}x_ix_j + \sum_i 2p_ix_i + \sum_i q_i = c$$

for suitable constants a_i, $b_{i,j}$, p_i, q_i and c. Many calculus textbooks (including the course text) describe a short list of standard examples in great detail and either suggest or assert that **ALL** solution sets for quadratic polynomials can be transformed into the standard examples by a suitable change of variables. This fact is a consequence of a basic result in linear algebra called *The Fundamental Theorem on Real Symmetric Matrices*. A detailed account of this fact appears in Section V.2 of the online file

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

and is beyond the scope of this course. However, we shall describe the main points in the classification here.

The crucial input from linear algebra can be summarized as follows:

RIGID CHANGES OF COORDINATES. *Suppose we are given a quadric surface or conic section Σ satisfying an equation as described above such that at least some of the second degree terms are nonzero. Then there exists a rectangular coordinate system $w_1, \cdots, w_n$ in which the defining equation has one of the following forms, in which $r \leq n$ and all coefficients except possibly c'' are nonzero.*

$$\sum_{j=1}^{r} d_j w_j^2 + c'' \qquad \sum_{j=1}^{r} d_j w_j^2 + k\, w_{r+1}$$

If $r = n$, it is understood that the second possibility does not arise.

The TYPE of the quadric or conic is given by the numbers of positive and negative coefficients among the constants d_j, the question of whether there is a nonzero first degree term in the equation, and the sign of c'', which is either positive, negative or zero.

If we make a change of variables of the form $u_i = G_iw_i + H_i$, where each G_i is nonzero, then the type of the quadric or conic does not change. This is useful because it allows us to modify the defining equation such that each d_j is ± 1, and the constant c'' lies in $\{-1, 0\}$. We can clearly rearrange the variables so that the positive d_j's precede the negative ones.

In order to write down the classification by standard types, one more step is needed. Namely, we need to let r_+ denote the number of positive d_j's. Then the standard forms we have obtained for quadrics in $\mathbf{R}^n$ can be classified in terms of the number r of nonzero d_j's, the number r_+ of positive d_j's, and the constant c' for equations of Type I. If we divide our quadratic polynomial by a nonzero constant, this will have no effect on the set of solutions to the polynomial, and therefore we can always arrange things so that the constant is equal to either -1 or 0. This means that we can consolidate things further in terms of the numbers of positive and nonzero d_j's for the second degree part of the polynomial and three possibilities for the form of the the nonquadratic part; namely, it is either a constant times some coordinate, it is equal to -1 or it is equal to zero. In dimensions 2 and 3 these classifications can be summarized by tables as follows:

STANDARD FORMS FOR CONICS IN THE PLANE

r	r_+	NONQUADRATIC PART	TYPICAL EXAMPLE	STANDARD DESCRIPTION
2	2	-1	$x^2 + y^2 = 1$	**ellipse**
2	2	0	$x^2 + y^2 = 0$	*one point*
2	1	-1	$x^2 - y^2 = 1$	**hyperbola**
2	1	0	$x^2 - y^2 = 0$	*pair of intersecting lines*
2	0	-1	$x^2 + y^2 = -1$	*no points*
2	0	0	$x^2 + y^2 = 0$	*one point*
1	1	linear	$x^2 = y$	**parabola**
1	1	-1	$x^2 = 1$	*pair of parallel lines*
1	1	0	$x^2 = 0$	*one line*
1	0	linear	$x^2 = y$	**parabola**
1	0	-1	$x^2 = -1$	*no points*
1	0	0	$x^2 = 0$	*one line*

Nondegenerate examples not consisting of one or two lines or one or zero points are designated by **boldface** type, and the remaining examples are designated using *italic* type.■

Note that many lines in the table deal with degenerate situations where the conic reduces to a one or two lines, a point, or the empty set. The corresponding table in the three-dimensional case appear below. As before, nondegenerate examples are in **boldface**.

STANDARD FORMS FOR QUADRICS IN 3-SPACE

r	r_+	NONQUADRATIC PART	TYPICAL EXAMPLE	STANDARD DESCRIPTION
3	3	-1	$x^2 + y^2 + z^2 = 1$	**ellipsoid**
3	3	0	$x^2 + y^2 + z^2 = 0$	*one point*
3	2	-1	$x^2 + y^2 - z^2 = 1$	**one-sheeted hyperboloid**
3	2	0	$x^2 + y^2 - z^2 = 0$	**elliptic cone**
3	1	-1	$x^2 - y^2 - z^2 = 1$	**two-sheeted hyperboloid**
3	1	0	$x^2 - y^2 - z^2 = 0$	**elliptic cone**
3	0	-1	$x^2 + y^2 + z^2 = -1$	*no points*
3	0	0	$x^2 + y^2 + z^2 = 0$	*one point*
2	2	linear	$x^2 + y^2 = z$	**elliptic paraboloid**
2	2	-1	$x^2 + y^2 = 1$	**elliptic cylinder**
2	2	0	$x^2 + y^2 = 0$	*line*
2	1	linear	$x^2 - y^2 = z$	**hyperbolic paraboloid**
2	1	-1	$x^2 - y^2 = 1$	**hyperbolic cylinder**
2	1	0	$x^2 - y^2 = 0$	*pair of intersecting planes*
2	0	linear	$x^2 + y^2 = z$	**elliptic paraboloid**
2	0	-1	$x^2 + y^2 = -1$	*no points*
2	0	0	$x^2 + y^2 = 0$	*one line*
1	1	linear	$x^2 + y^2 = z$	**parabolic cylinder**
1	1	-1	$x^2 = 1$	*pair of parallel planes*
1	1	0	$x^2 = 0$	*one plane*
1	0	linear	$x^2 = y$	**parabolic cylinder**
1	0	-1	$x^2 = -1$	*no points*
1	0	0	$x^2 = 0$	*one plane*