Vector fields, flow curves and the ∇ – operator

Although the mathematical definitions of a ***vector field***, it ***divergence*** and its ***curl*** are relatively straightforward, the reasons for introducing these concepts are probably less apparent. We have already noted that vector fields arise in various physical contexts as ***force fields*** or ***velocity fields***, and we shall begin with some further comments on the ***flow curves*** of a vector field.

Let $\mathbf{U}$ be an open region in the coordinate plane or 3 – space, and suppose that $\mathbf{F}$ is a vector field on $\mathbf{U}$ whose coordinate functions (P, Q) or (P, Q, R) all have continuous second derivatives; one can get by with weaker assumptions at several points, but if we assume the given conditions this will eliminate some possible distractions. Suppose now that $\mathbf{U}$ is filled with some sort of fluid and $\mathbf{F}$ defines a velocity field such that if we are given a point $\mathbf{r} = (a, b, c)$ in the fluid, then its velocity at $\mathbf{r}$ is equal to $\mathbf{F}(a, b, c)$; if we are working with only two dimensions we shall simply assume that $c = 0$ and drop the final coordinate. Our physical intuition suggests that for each $\mathbf{r}$ the fluid motion defines a reasonable curve $s(t)$, defined for all times t sufficiently close to time 0, such that $s(0) = \mathbf{r}$ and $s'(t) = \mathbf{F}(s(t))$ for such values of t. Mathematically this corresponds to the existence of solutions to the system of differential equations

$$x' = P(x, y, z), \quad y' = Q(x, y, z), \quad z' = R(x, y, z)$$

(where the data involving z, R and z' appear only in the 3 – dimensional case)

such that $x(0) = a$, $y(0) = b$ and $z(0) = c$. Standard existence and local uniqueness results on differential equations guarantee the mathematical existence of solutions to such systems of differential equations.

Using the flow curves we can discuss the ***dynamical behavior*** of the fluid. Specifically, if we take a small disk or solid rectangular box containing a point, then for small enough values of t the molecules which are inside that disk or box will all move with the passage of time, and the motion will transform the original figure into some other figure in the open region $\mathbf{U}$. Several animated and interactive examples of this are given in the following online site:

http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Flow/flow.html

If one clicks on the examples from this site, it becomes apparent that the sizes and shapes of figures can become extremely distorted under the motion of the fluid. The divergence of the vector field gives some first order information on this distortion; namely, for sufficiently small disks or boxes centered at the given point $\mathbf{r}$, the divergence of $\mathbf{F}$ at $\mathbf{r}$ gives a first order estimate of the rate at which the area or volume of the figure is changing when $t = 0$ (the volume/area is increasing if the divergence is positive, decreasing if it is negative, and if the divergence is zero this indicates that the volume/area is more or less constant under the motion of the fluid – this is the reason why a vector field is said to be ***incompressible*** if its divergence is always zero). Here are three sites with further discussion of the divergence and its physical interpretation:

http://www.youtube.com/watch?v=JAXyLhvZ-Vg&feature=PlayList&p=19E79A0638C8D449&index=82

http://www.youtube.com/watch?v=tOX3RkH2guE&feature=PlayList&p=19E79A0638C8D449&index=83

http://www.youtube.com/watch?v=U6Re4xT0o4w&feature=PlayList&p=19E79A0638C8D449&index=84

A physical interpretation for the curl of a vector field is somewhat more elusive. If we take our previous interpretation of the vector field $\mathbf{F}$ as the velocity field for a fluid in motion, then one can view the curl as a measure of how the disk or box is rotating around the flow curve for the given reference point. The following interactive sites with animation contain good illustrations of such phenomena.

http://info.ee.surrey.ac.uk/Teaching/Courses/EFT/dynamics/html/curl.html

http://matematicas.uniandes.edu.co/~jarteaga/proyectos/coord-calvec/archivos/thomas/java/mod6_test.html

Here is a sequence of sites which provides further discussion of the curl and its physical interpretation; these videos and the preceding ones on the divergence are part of a longer series.

http://www.youtube.com/watch?v=Mt4dpGFVsYc&feature=PlayList&p=19E79A0638C8D449&index=85

http://www.youtube.com/watch?v=hTSyVgBa1T0&feature=PlayList&p=19E79A0638C8D449&index=86

http://www.youtube.com/watch?v=fYzoiWIBjP8&feature=PlayList&p=19E79A0638C8D449&index=87

Finally, here are two more online references. The first leads to more files like the immediately preceding ones, covering a very wide range of topics in calculus. The second is interactive and animated, and it contains more examples of fluid – like flows associated to a velocity field.

http://www.youtube.com/user/khanacademy

http://publicliterature.org/tools/vector_field/