Functions with vanishing partial derivatives

The Mean Value Theorem implies that a differentiable function on an interval is constant if and only if its derivative is always zero. If a function of n variables is constant on some open neighborhood defined by an inequality of the form $|\mathbf{v} - \mathbf{a}| < h$, where $h > 0$, then it follows that all the partial derivatives of f are zero. We can derive the converse of this fact from the multivariable chain rule and the validity of the result for functions of one variable:

THEOREM. *Let U be an open neighborhood as in the preceding discussion, and assume that the real valued function f is defined on U such that it has continuous first partial derivatives everywhere and all these partial derivatives are equal to zero. Then f is constant.*

Proof. Let $\mathbf{a}$ be the center point of U, and let $\mathbf{x} \in U$. Define $\gamma(t) = \mathbf{x} + t(\mathbf{x} - \mathbf{a})$, and let $h(t) = f\big(\gamma(t)\big)$. By the Chain Rule for partial differentiation we have

$$h'(t) \quad = \quad \nabla f\big(\gamma(t)\big) \cdot \gamma'(t)$$

and since $\nabla f = \mathbf{0}$ it follows that $h' = 0$, and by the known result for functions of one variable, this means that h is constant. Therefore we have $f(\mathbf{x}) = h(1) = h(0) = f(\mathbf{a})$. Since $\mathbf{x}$ was arbitrary this means that f is constant on U.■

The same argument would apply if we took U to be the open square/cube/hypercube consisting of all $\mathbf{x}$ such that $a_i - h < x_i < a_i + h$, where the symbols a_i and x_i denote the i^{th} coordinates of $\mathbf{a}$ and $\mathbf{x}$ respectively.