Comments on Colley, Section 5.1

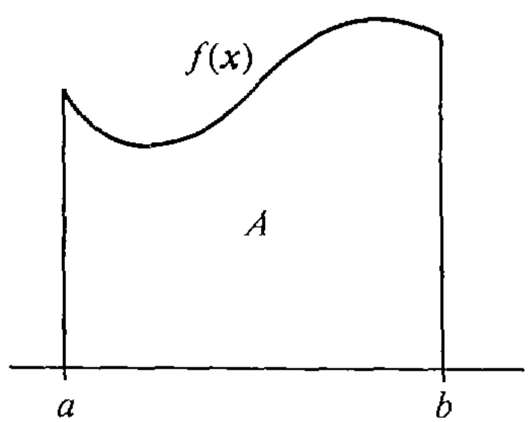
The purpose of this section is to motivate the concept of a multiple integral, and the discussion below is designed to include some additional background. In this document we have used several drawings from the following online documents; however, the treatment in those documents is much different from the approach taken here.

http://www.math.wisc.edu/~keisler/chapter_12.pdf

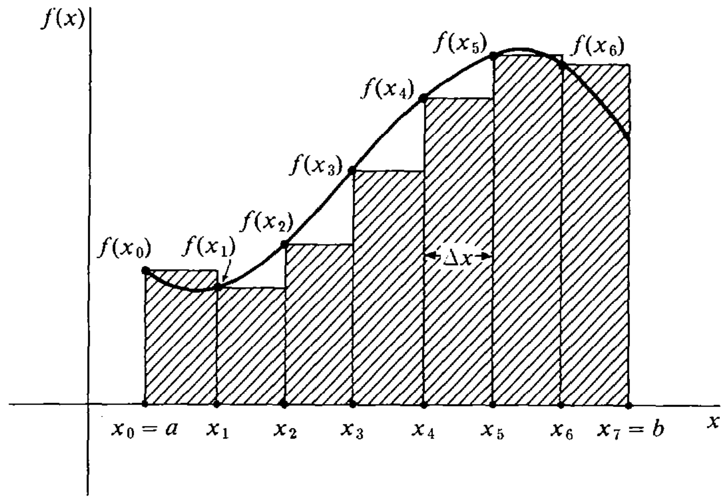
http://www.math.wisc.edu/~keisler/chapter_4.pdf

For most if not all topics in multivariable calculus, it is useful to begin by reviewing the corresponding material in single variable calculus, and integration is one example of this principle. Therefore we start with a quick review of definite integrals.

One motivation for the definite integral is the problem of finding the areas of regions bounded by curves. Some of the simplest examples are given by the regions bounded by the x – axis, the vertical lines $x = a$ and $x = b$ (as usual, we assume $a < b$), and the graph of some function $y = f(x)$, where f is a continuous and positive valued function on the interval $[a, b]$. A typical example is sketched below



The definition of the definite integral of $f(x)$ from $x = a$ to $x = b$ reflects a simple principle: One can approximate the area bounded by these curves by a finite collection of rectangles which geometrically approximate the pieces of the region A whose x – coordinates lie in suitable subintervals of $[a, b]$.

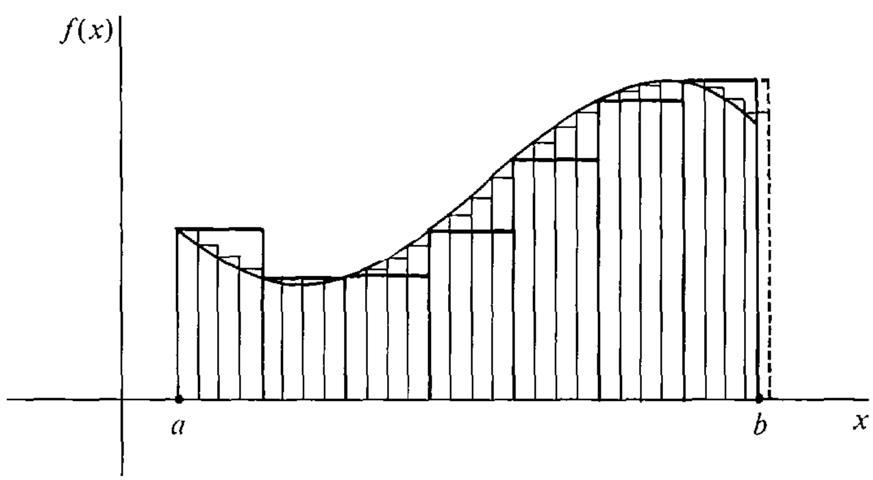


The shaded region in the drawing above is an approximation to the area bounded by the original curves, and it is often written in a form resembling

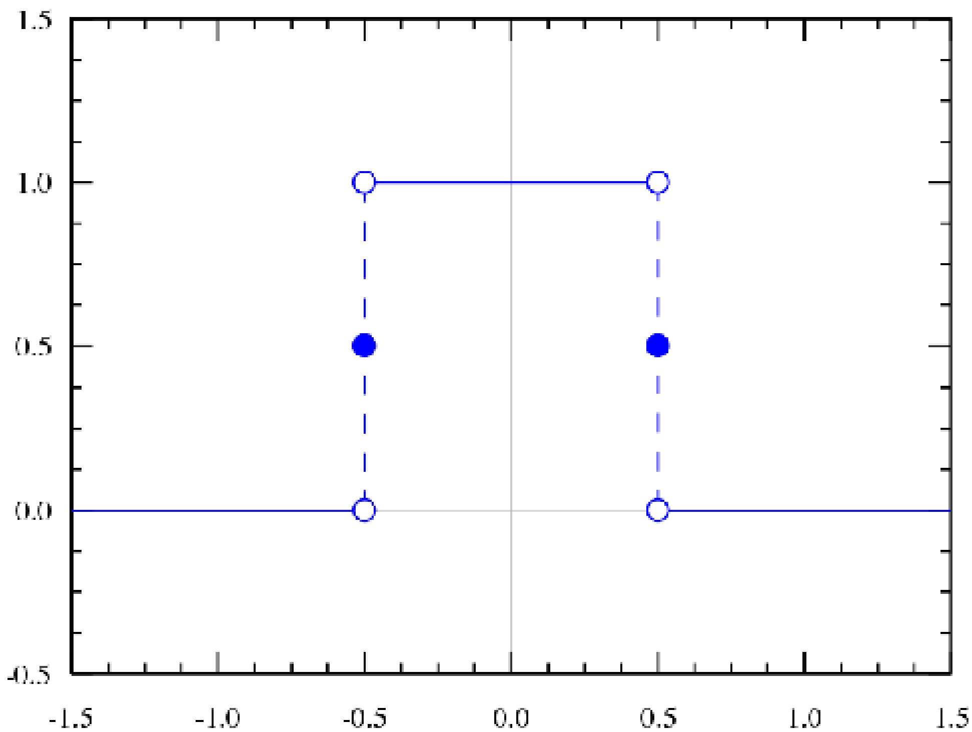
$$\sum_{a}^{b} f(x)\,\Delta x = S(a, b, \Delta x).$$

Each summand on the left hand side represents the area of one rectangle in the collection. The width of such a rectangle is $\boldsymbol{\Delta x}$ and the height is $\boldsymbol{f}\,(\boldsymbol{U})$ for some $\boldsymbol{U}$ in the associated interval on the $\boldsymbol{x}$ – axis.

Given one approximation to the area, the natural next question is to find even better approximations. Experience suggests that we can do this by taking more rectangles such that their widths are smaller than the widths for the original approximation (see the drawing below).



If we continue this process, it is reasonable to ask whether the limit of the approximation sequence will be the area of the region bounded by the original curves. This is true for most of the functions which arise in the sciences and their applications, but in order to justify this hypothesis it is necessary to give a formal proof; the latter is generally given in an undergraduate real variables course, and at this point all we need to know is that there is a well – behaved limit equal to the area if the function f is continuous on the interval or at least continuous "almost everywhere" on the interval (for example, there might be finitely many jump discontinuities as in the drawing below); in the area problem the function ***f*** is assumed to take positive values, but it turns out that all the mathematical discussion works regardless of whether or not the function's values are all positive. We note the possibility of discontinuities because they play a particularly important role in the description of multiple integrals.

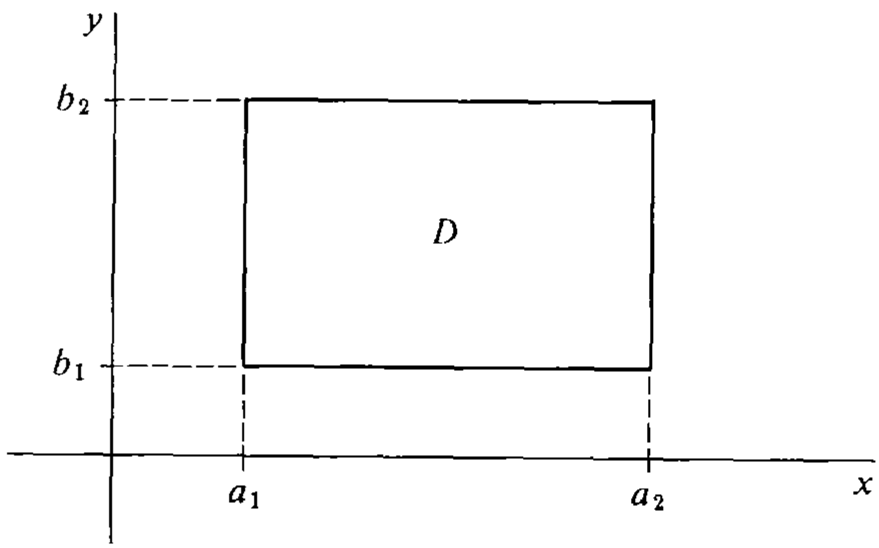


(**Source:** **http://en.wikipedia.org/wiki/File:Rectangular_function.svg**)

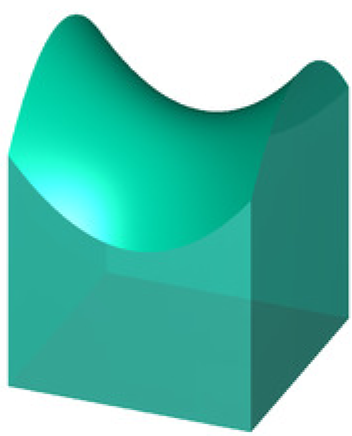
In this example, the function has jump discontinuities at $x = \pm \frac{1}{2}$.

Extension to functions of two variables

If integration of functions in one variable corresponds to finding areas, then it is natural to guess that integration of functions in two variables corresponds to volumes. This is indeed the case, and one approach to defining the integral of a function $f(x, y)$ of two variables is to view the integral as the volume of a particular solid. For example if the function is continuous, positive valued, and defined on some solid rectangle D given by inequalities of the form $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$ (see the drawing),



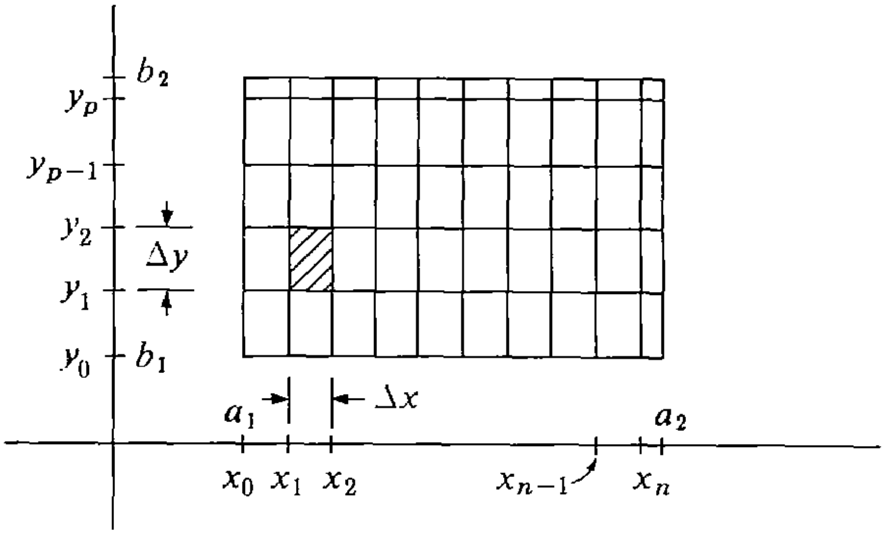
then the integral of f over D should correspond to the volume of the solid bounded by the xy – plane, the planes $x = a_1, x = a_2, x = b_1, x = b_2$, and the graph of f.



Double integral as volume under the surface $z = 9 + x^2 - y^2$, where $|x|, |y| \leq 2$. The rectangular region at the bottom of the body is the domain of integration, while the upper surface is the graph of the two – variable function to be integrated.

(**Source:** **http://en.wikipedia.org/wiki/Multiple_integral**)

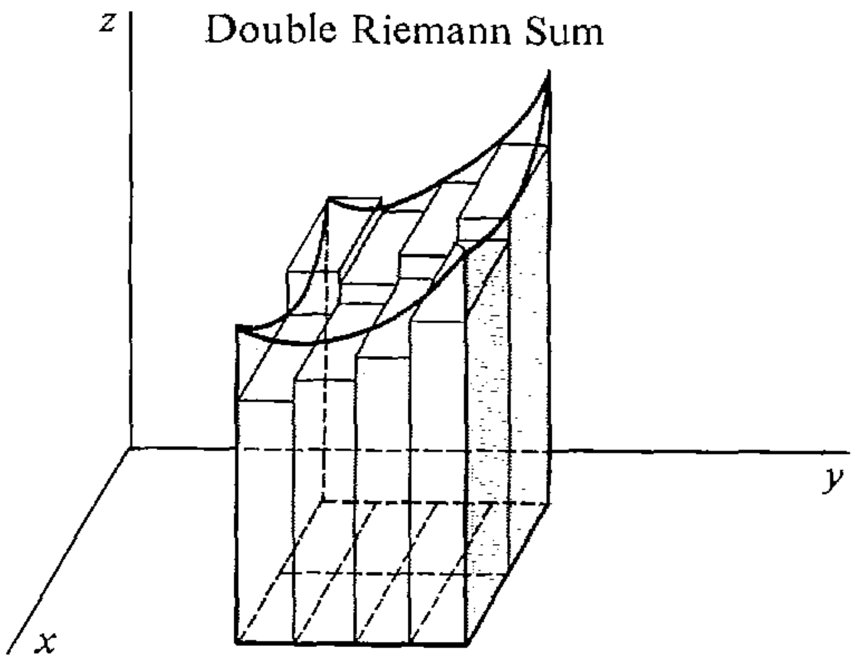
Specifically, we proceed as follows: First, we partition the original solid rectangular region D into a finite subcolletion of smaller, nonoverlapping solid rectangular regions as in the drawing below:



As suggested by the discussion above and the drawing below, a ***double Riemann sum*** of the form

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y$$

approximates the volume of the solid over D between $z = 0$ and $z = f(x, y)$.



If we partition each of the small solid rectangular regions into even smaller small solid rectangular regions, then as before it should seem likely that the Riemann sum approximation to the volume will improve, and in fact one can prove this rigorously. Furthermore, one might guess that if one takes a limit of Riemann sum approximations as the dimensions of the small rectangles go to zero in some reasonable way, then the values of these approximations tend to a limit value which is the volume of the original solid. Once again, it is necessary to justify this theoretically by proving that a common limit value actually exists, but the details are beyond the scope of this course. For our purposes it will suffice to know that the limit exists provided the function is continuous "almost everywhere." In the single variable case, the most basic examples of the latter were finite collections of points, but in the two – variable case one must also consider the points on finite collections of well – behaved curves (piecewise continuously differentiable are included in this).

Additional online references

We shall begin with some general references, including a site which covers both this course as well as its prerequisite, a site with further background material, and a site discussing common mathematical errors.

http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

http://tutorial.math.lamar.edu/Extras/AlgebraTrigReview/AlgebraTrigIntro.aspx

http://tutorial.math.lamar.edu/Extras/CommonErrors/CommonMathErrors.aspx

The following items from the first site are particularly relevant to the material discussed above:

http://tutorial.math.lamar.edu/Classes/CalcIII/DoubleIntegrals.aspx

http://tutorial.math.lamar.edu/Classes/CalcIII/DIGeneralRegion.aspx

The two sites listed below also contain numerous graphical examples, but the pictures appear and disappear rather quickly. In order to examine them in a more leisurely fashion it is necessary to download the Mathematical Player software, for which a link is given.

http://demonstrations.wolfram.com/RiemannSumsForFunctionsOfTwoVariables/

http://demonstrations.wolfram.com/DoubleIntegralForVolume/

http://www.wolfram.com/products/player/download.cgi

Finally, here is one more site with animated drawings which are related to the preceding discussion.

http://www.math.ou.edu/~tjmurphy/Teaching/2443/DoubleIntegrals/doubleIntegrals.html

Most of the sites listed above are also relevant to the material in the next section of the course text.