Comments on Colley, Section 5.2

This section goes further into the properties of double integrals. As before, we have used several drawings and displays from the following online document, but the treatment of material is much different here.

http://www.math.wisc.edu/~keisler/chapter_12.pdf

Several basic properties of double integrals are more or less to be expected, but they are important enough to be noted explicitly. As before, we assume that we are integrating over a solid rectangular region ***D*** that is suitably placed with respect to the coordinate axes. The first property is very simple to state:

*If **f(x,y)** has a constant value **c** over **D,** then its integral is **c** times the area of **D.***

Geometrically, this just reflects the principle that the volume of a right cylinder with base ***D*** and height ***c*** is equal to the given value.

Here are some further properties:

CONSTANT RULE

$$\iint_D cf(x, y)\, dA = c\iint_D f(x, y)\, dA.$$

SUM RULE

$$\iint_D f(x, y) + g(x, y)\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA.$$

INEQUALITY RULE

If $f(x, y) \le g(x, y)$ *for all* (x, y) *in* D,

$$\iint_D f(x, y)\, dA \le \iint_D g(x, y)\, dA.$$

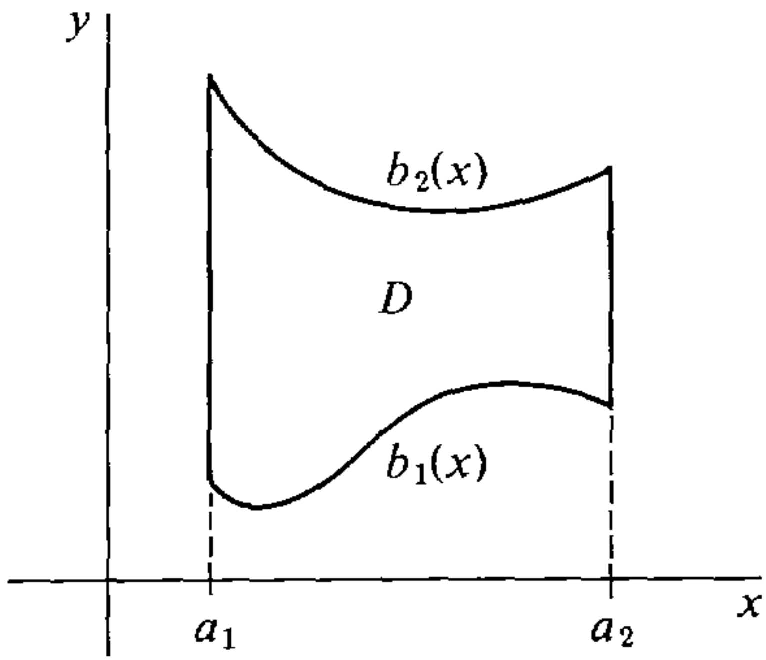
The latter property leads to the following important principle for estimating the values of double integrals.

CYLINDER PROPERTY

Let m and M be the minimum and maximum values of $f(x, y)$ on D and let A be the area of D. Then

$$mA \leq \iint_D f(x, y)\, dA \leq MA.$$

There is one more property to state, but before doing so we shall introduce another main point in this section; namely, the definition of double integrals for regions ***D*** that are not rectangular. The idea is simple. At this point we are only considering regions ***D*** that are ***bounded*** in the sense that the coordinates of all points in ***D*** are bounded, so that there is some large solid rectangular region ***E*** which contains D. Given a function f on ***D***, define an extended function f_E such that f_E is zero on all points which do not lie on ***D***. Then the integral of ***f*** over ***D*** is equal to the integral of ***f*** over f_E.



Once again it is necessary to use some logic in order to justify this definition and to show that it works in the basic cases of interest. For example, we need to check that the value will not depend on the particular choice of rectangular region ***E***. More important, we also need to check that a suitable limit exists for cases which arise frequently in the sciences and their applications. One recurrent example is illustrated above; the region ***D*** is bounded by the two vertical lines and the graphs of the functions $\boldsymbol{y = b_1(x)}$ and $\boldsymbol{y = b_2(x)}$. In this case the justification is given by the following fine print.

> If we have a continuous function on $\boldsymbol{D}$ and extend it to a continuous function on some larger rectangular region $\boldsymbol{E}$, then the discontinuities of the function will all lie on the boundary curves of $\boldsymbol{D}$, so the extended function is continuous almost everywhere, and we have already noted that the integral exists if the function is continuous almost everywhere on a rectangular region.

With these conventions, all the preceding properties of double integrals go through for reasonable closed regions $\boldsymbol{D}$. Furthermore, we also have the following crucial fact:

ADDITION PROPERTY

Let D be divided into two regions D_1 and D_2 which meet only on a common boundary as in Figure 12.1.13. Then

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA.$$

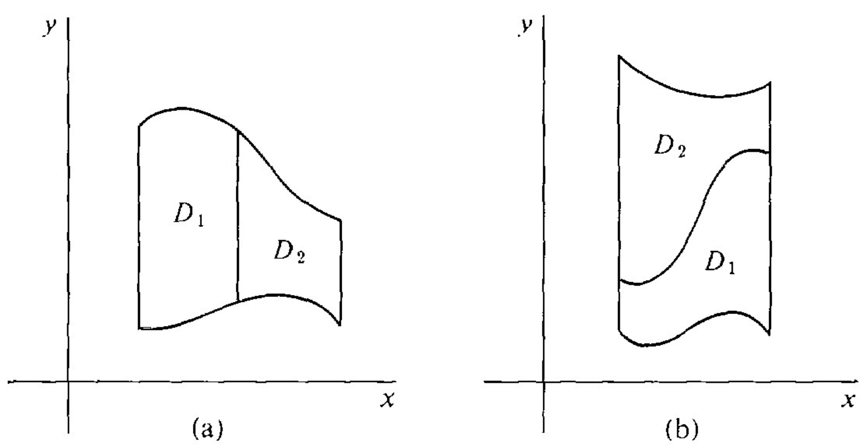


Figure 12.1.13

Computing double integrals

None of the definitions or properties of double integrals are necessarily useful unless we have some way of computing them, at least in important cases. In ordinary multivariable calculus, the Fundamental Theorem of Calculus is an extremely powerful tool for computing ordinary integrals, and we need some way of "leveraging" this into a comparable tool for double integrals. The means for doing this is given by ***iterated integrals***, and the main result is given below. In effect, it reduces the computation of double integrals to the computation of two ordinary integrals if D is given as before as a region bounded by the two vertical lines $x = a_1$ and $x = a_2$ along with the graphs of the functions $y = b_1(x)$ and $y = b_2(x)$.

ITERATED INTEGRAL THEOREM

Let D be a region

$$a_1 \leq x \leq a_2, \qquad b_1(x) \leq y \leq b_2(x).$$

The double integral over D is equal to the iterated integral:

$$\iint_D f(x, y)\, dA = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f(x, y)\, dy\, dx.$$

The basic idea is to begin by integrating the inside integral with respect to y, with x viewed as a constant. Since the upper and lower limits of the inside integral are given in terms of x, the inside integral will be some function of x. We can then (try to) integrate this function with respect to x in any of the usual ways to obtain a precise numerical value. Obviously this works well in many cases; for example, there will be no problems if the functions $y = b_1(x)$ and $y = b_2(x)$ are polynomials in x and $f(x, y)$ is a polynomial in x and y. Animation in the previously cited file

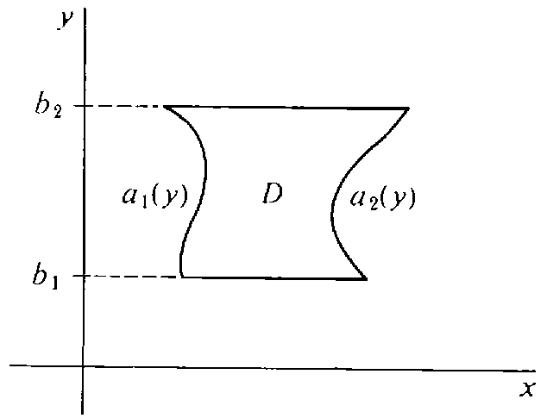
http://www.math.ou.edu/~tjmurphy/Teaching/2443/DoubleIntegrals/doubleIntegrals.html

motivates and illustrate the Iterated Integral Theorem.

Many (most?) of the homework problems in Sections 5.1 and 5.2 involve special cases of the Iterated Integral Theorem with specific choices for the functions $b_1(x)$, $b_2(x)$ and $f(x, y)$. In some cases, the regions are not given explicitly in the nice form and it is necessary to do some work in order to find the limits of integration. For example, the region D might be given as the one bounded by the line $y = 2x$ and the parabola $y = x^2 - 3$. In this case one would begin by sketching the region (**do it now!**), and noticing that **(1)** the two curves meet at a pair of points with different first coordinates, **(2)** between these two coordinates the line $y = 2x$ is above the parabola $y = x^2 - 3$. This immediately gives us $b_1(x)$ and $b_2(x)$, so all we need to do is find the limits of integration with respect to x. These are given by the solutions to the quadratic equation $2x = x^2 - 3$, whose roots are given by **3** and **– 1.** Therefore, **– 1** and **3** are the desired limits of integration with respect to x in this example.

Further cases

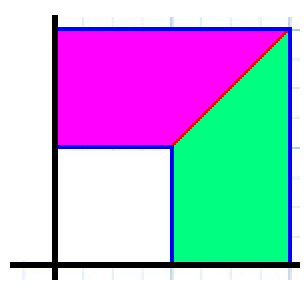
Everything above will go through if we interchange the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$, in which case the region $\boldsymbol{D}$ is bounded by the two vertical lines $\boldsymbol{y = b_1}$ and $\boldsymbol{y = b_2}$ along with the graphs of the functions $\boldsymbol{x = a_1(y)}$ and $\boldsymbol{x = a_2(x)}$**.** A typical region of this type is illustrated below:



For such an example, the double integral is given by an iterated integral such that the inside term is formed by integration with respect to $\boldsymbol{x}$ and the outside term is formed by intersection with respect to $\boldsymbol{y}$.

$$\iint_D f(x, y)\, dA = \int_{b_1}^{b_2} \int_{a_1(y)}^{a_2(y)} f(x, y)\, dx\, dy.$$

Of course, there are many regions which cannot be expressed in either of the terms we have described. One example, illustrated below, is the set of all points in the coordinate plane $(\boldsymbol{x}, \boldsymbol{y})$ such that either **(*i*)** $\boldsymbol{1 \leq x \leq 2}$ and $\boldsymbol{0 \leq y \leq 2}$ or else **(*ii*)** $\boldsymbol{0 \leq y \leq 2}$ or $\boldsymbol{1 \leq x \leq 2}$**.**



This region $\boldsymbol{D}$ splits into two regions $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ along the curve $\mathbf{y = x}$, where one has the form $\boldsymbol{1 \leq x \leq 2}$ and $\boldsymbol{0 \leq y \leq x}$ (shaded in green), and the other has the form $\boldsymbol{1 \leq y \leq 2}$ and $\boldsymbol{0 \leq x \leq y}$ (shaded in pink). One can integrate over $\boldsymbol{D}$ by using this decomposition of $\boldsymbol{D}$ into $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ together with the Addition Formula and the Iterated Integral Theorem(s).

Most of the reasonable closed regions encountered in undergraduate mathematics or the sciences have finite decompositions into nonoverlapping pieces for which one of the iterated integral formulas is valid (see Widder, ***Advanced Calculus***, p. 225, for a reference and example).

Additional references

In the preceding discussion we mentioned the following classic text:

D. V. Widder. ***Advanced Calculus.*** Dover, 1989.

This book is written at a somewhat more advanced level than the course text, but it often has more complete information on certain points. The following outline/review book is also recommended to those who wish to see things from a viewpoint related to these notes but with more details:

Colin Adams, Abigail Thompson and Joel Hass. ***How to Ace the Rest of Calculus: The Streetwise Guide, Including Multi-Variable Calculus.*** Freeman, 2001.

Partial previews of this book are available on the World Wide Web.

The following online item from a previously noted site is also relevant to this section:

http://tutorial.math.lamar.edu/Classes/CalcIII/DIGeneralRegion.aspx