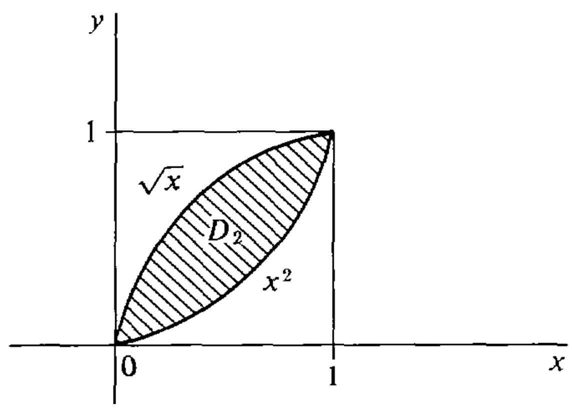
Comments on Colley, Section 5.3

Interchanging the order of integration is a standard technique in multivariable calculus, and problems on this topic appear in the exams for most if not all multivariable calculus courses. Some illustrations below are taken from a previously cited reference: **http://www.math.wisc.edu/~keisler/chapter_12.pdf**.

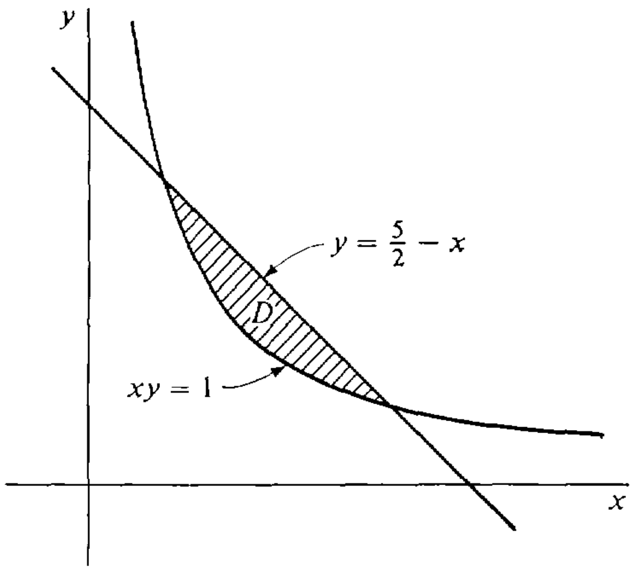
The basic idea is simple. Suppose that we are given a region D which can be described as the region between two graphs $y = g(x)$, $y = h(x)$, where g and h are ***strictly increasing*** functions defined on the interval $[a, b]$ which satisfy $g \leq h$, and for the sake of convenience let us also assume that the values of g and h at the endpoints a and b are equal. For example, over the unit interval $[0, 1]$ we can take $g(x) = x^r$ and $h(x) = x^s$, where $0 < s < r$. A typical example is illustrated below.



In the drawing it is clear that one can also describe the shaded region as the one bounded by the graphs of $x = y^2$ and $x = y^{1/2}$. Therefore there are two different ways of expressing the integral of the shaded region as an iterated integral: One can integrate first with respect to y and then with respect to x or vice versa. More generally, if g and h are as in the preceding paragraph and we set $c = g(a) = f(a)$ and $d = g(b) = f(b)$, suppose that p and q are the inverse functions to f and g; in other words, $x = p(y)$ if and only if $y = g(x)$, and $x = q(y)$ if and only if $y = h(x)$. Then the region defined by the conditions $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ is also defined by the conditions $c \leq y \leq d$ and $q(y) \leq x \leq p(y)$. Notice the placement of p and q in the inequality; when one takes the inverse functions it is necessary to ***reverse*** the directions of inequalities.

Example. Suppose we modify the drawing above, taking instead the graphs of the functions $y = x^2$ and $y = x^{1/3}$ (so the first graph lies below the second over the interval $[0, 1]$). This region can also be described as the region bounded by the lower curve $x = y^3$ and $x = y^{1/2}$.

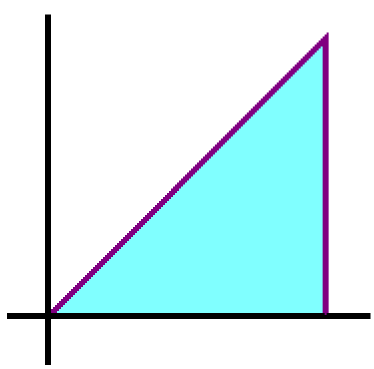
A similar principle holds if g and h are both ***strictly decreasing*****;** in this case the directions of the inequalities for the inverse functions do **not** change. Here is an example with a curious additional property: Each of the functions g and h is equal to its own inverse. The coordinates of the intersection points for these curves are $(\frac{1}{2}, 2)$ and $(2, \frac{1}{2})$; one finds these values by solving $g(x) = h(x)$.



A standard interchange formula (sometimes called Dirichlet's formula)

(The name is pronounced DEER – i – shlay.)

There are also other cases in which one can interchange the order of integration. One particular example is the solid triangle defined by the inequalities $0 \le x \le 1$ and $0 \le y \le x$ (the shaded region in the drawing below).



This region is also describable by the conditions $0 \le y \le 1$ and $y \le x \le 1$, and therefore we have the following integral identity.

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x,y)\,dy\,dx = \int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x,y)\,dx\,dy$$

(**Source: http://calculuspowerup.com/the-order-of-integration-and-fubinis-theorem/**)

One should compare this result to the true – false questions 5.7.5 and 5.7.7 on page 357 of Colley and determine whether the formulas in those questions are correct. Notice that the formula is not symmetric in the ***x*** and ***y*** variables.

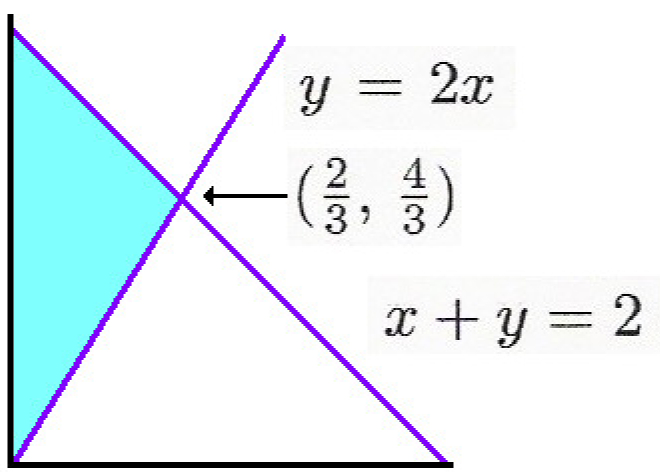
The first step in working any problem on interchanging the order of integration is to draw a good picture and to determine important visual features, like where various curves meet and where one curve is positioned with respect to the other(s).

Frequently one can use interchanging the order of integration to put a double integral into simpler terms. A typical problem of this sort is Example 2 on pages 310 – 311 of Colley.

Here is an example in which interchanging the order of integration can make things more complicated:

Set up iterated integrals for computing the double integral of $f(x, y)$ over the region bounded by the lines $y = 2x$, $x + y = 2$ and the $y-$axis.

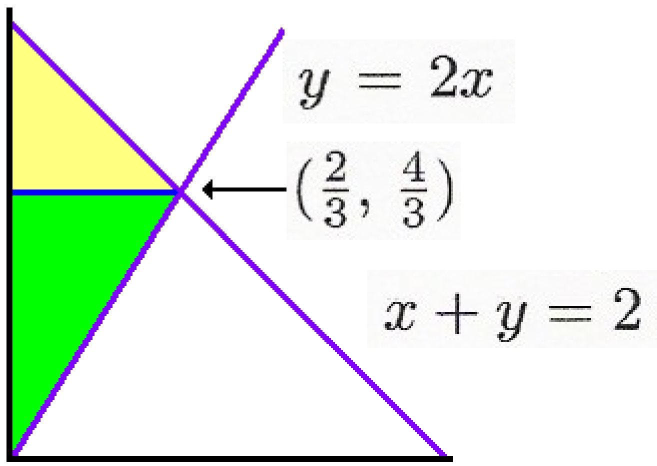
The first step in solving this exercise is to sketch the region, which is a solid triangle whose vertices are the origin, the point $(0, 2)$, and the point $(\frac{2}{3}, \frac{4}{3})$.



One sees that the x limits of integration are between 0 and $\frac{2}{3}$, while the y limits are between the graphs of $y = 2x$ on the bottom and $y = 2 - x$ on the top. Thus it is straightforward to see that the double integral is equal to the following iterated integral:

$$\int_0^{2/3} \int_{2x}^{2-x} f(x, y)\, dy\, dx$$

Now suppose we want to integrate in the opposite order. In this case we need to split the region into two pieces such that the limits of integration over each piece are simply described as graphs of functions $x = g(y)$. The splitting takes place along the horizontal line containing the point where the graphs of $2x$ and $2 - x$ meet — we already know this point has coordinates $(\frac{2}{3}, \frac{4}{3})$.



The upper piece is bounded by the curves $y = \frac{4}{3}$, $y = 2$, $x = 0$, and $x = 2 - y$, while the lower piece is bounded by the curves $y = \frac{4}{3}$, $y = 0$, $x = 0$, and $x = \frac{1}{2}y$. Therefore if we want to express the original double integral using iterated integrals in the reverse order, we obtain the following:

$$\int_0^{4/3} \int_0^{y/2} f(x, y)\, dx\, dy \quad + \quad \int_{4/3}^{2} \int_0^{2-y} f(x, y)\, dx\, dy$$