Comments on Colley, Section 5.4

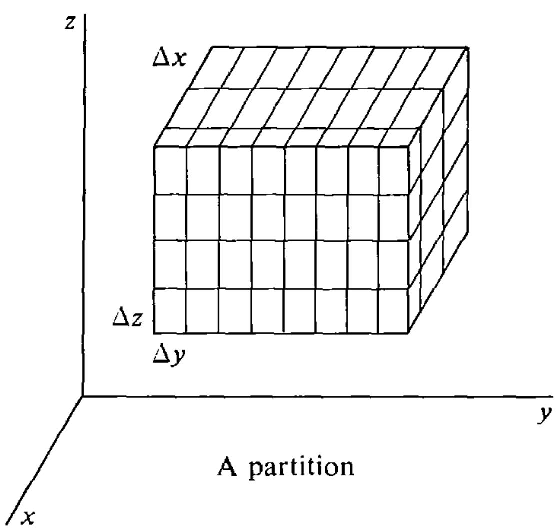
In the first course in this sequence there was a natural progression from functions of one variable to functions of two and three variables, and ultimately even more variables. Similarly, there is a natural progression from ordinary integration and double integration to triple and even higher order multiple integrations. In this course we shall consider triple integration only because it is used in many branches of the sciences and engineering, while quadruple and higher order integrals appear only in more advanced contexts. Once again we shall use illustrations from the online notes **http://www.math.wisc.edu/~keisler/chapter_12.pdf** at various points.

We began the discussion of double integrals with a problem from physics that illustrated the need for such integrals, and we shall proceed similarly for triple integrals. Suppose that we are given a rectangular box which is filled by material(s) of variable density, and we wish to compute its mass. One response to such a problem is simply to weigh the object in question, but there are cases in which this is not possible and one needs some indirect method of determining the object's mass.

If the density of an object is constant, then one has the elementary formula

$$\textbf{\textit{mass}} \quad = \quad \textbf{\textit{density}} \times \textbf{\textit{volume}}$$

from high school science courses, and we shall use this to compute estimates for the mass of a rectangular box for which the variable density is given by some nonnegative valued continuous function $f(x, y, z)$. As in the case of double integrals, the first step is to partition the original rectangular box into smaller boxes over which the density does not fluctuate very much.



A partition

The mass of each smaller box is approximately equal to $f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k$, and the mass of the large box will be approximately equal to the sum of these terms. The mathematical name for such an expression is a ***triple Riemann sum*.**

$$\sum\sum_{E}\sum f(x, y, z)\,dx\,dy\,dz.$$

If we partition each of the small solid rectangular boxes into even smaller small solid rectangular boxes, then as before it should seem likely that the Riemann sum approximation to the mass will improve, and in fact one can prove this rigorously. Furthermore, just as before one might guess that if one takes a limit of Riemann sum approximations as the dimensions of the small boxes go to zero in some reasonable way, then the values of these approximations tend to a limit value which is the mass of the original solid. Once again, it is necessary to justify this theoretically by proving that a common limit value actually exists, but the details are beyond the scope of this course, and as before it will suffice to know that the limit exists provided the function is continuous "almost everywhere." In the three variable case, the "almost everywhere" condition is satisfied if the function is continuous off a finite collection of surfaces.

Triple integrals have many properties similar to those of double integrals. Here is one of the most important ones.

ITERATED INTEGRAL THEOREM

Suppose that **E** *is the rectangular box defined by the inequalities*

$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2, \qquad c_1 \leq z \leq c_2,$$

and let f be a continuous function defined on ***E.*** *Then the triple integral*

$$\iiint_{E} f(x, y, z)\,dV.$$

is equal to each of the following six iterated integrals:

$$(1) \quad \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx \qquad (2) \quad \int_{a_1}^{a_2}\int_{c_1}^{c_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dz\,dx$$

$$(3) \quad \int_{b_1}^{b_2}\int_{a_1}^{a_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dx\,dy \qquad (4) \quad \int_{b_1}^{b_2}\int_{c_1}^{c_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dz\,dy$$

$$(5) \quad \int_{c_1}^{c_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dx\,dz \qquad (6) \quad \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz.$$

EXAMPLE. Find the triple integral of the function $\sin(x+y+z)$ over the rectangular box defined by $0 \le x, y, z \le \pi/2$.

By the Iterated Integral Theorem we know that the triple integral is equal to the following iterated integral:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y+z)\,dz\,dy\,dx$$

Since the original data are symmetric in the three variables, it is also equal to the five iterated integrals obtained by suitably interchanging the roles of x, y and z, but we shall not use this fact.

Integrating first with respect to z, we see that the integral in question is equal to

$$\int_0^{\pi/2}\int_0^{\pi/2} -\cos(x+y+z)\Big|_{z=0}^{z=\pi/2}\,dy\,dx$$

and since $\cos(\theta+\frac{\pi}{2}) = -\sin\theta$ the triple integral is equal to the following double integral:

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)+\cos(x+y)\,dy\,dx$$

Next, we compute the double integral, and as before we obtain the following expression for it:

$$\int_0^{\pi/2} \sin(x+y)-\cos(x+y)\Big|_{y=0}^{y=\pi/2}\,dx$$

Now $\sin(\theta+\frac{\pi}{2}) = \cos\theta$, so we can rewrite the preceding as the following ordinary integral:

$$\int_0^{\pi/2} (\cos x+\sin x) - (\sin x-\cos x)\,dx = \int_0^{\pi/2} 2\cos x\,dx$$

and the latter is equal to

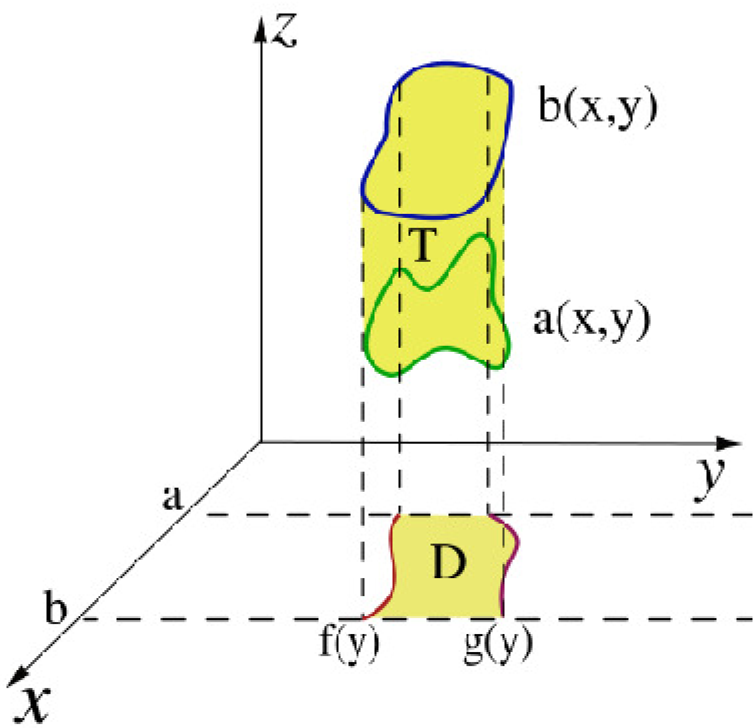
$$2\sin x\Big|_0^{\pi/2} = 2\,.$$

One can now define triple integrals over more general bounded regions by the same process used in the case of two variables: Extend the original function to some large rectangular box as above by setting it equal to zero off the original region, and then form the integral of this extended function. One can view this physically as expanding the original object $\boldsymbol{A}$ to a rectangular box in which there is no mass, and hence zero density, outside of $\boldsymbol{A}$.

There is a corresponding iterated integral formula for evaluating triple integrals over special types of elementary regions defined by inequalities as follows:

$$a \le x \le b, \quad f(x) \le y \le g(x), \quad a(x, y) \le z \le b(x, y)$$

A typical **3** – dimensional region of this type is illustrated below:



(Source: **http://en.wikipedia.org/wiki/Multiple_integral**)

Here is the formal statement that we want:

ITERATED INTEGRAL THEOREM

If E is the region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y),$$

then $$\iiint_E f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

Similar formulas hold if one interchanges the roles of x, y and z in the preceding discussion.

As in the **2** – dimensional case, one can use these formulas as one step in computing triple integrals over more general regions by splitting the latter into elementary regions, computing the integrals of the latter using the iterated integral formula(s), and finishing by applying the following identity:

ADDITION PROPERTY

If E is divided into two regions E_1 and E_2 which meet only on a common boundary then

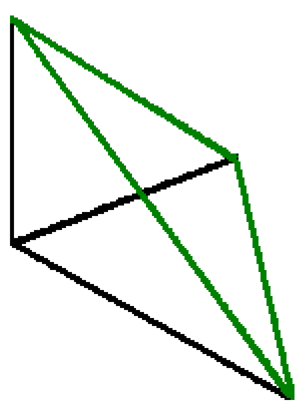
$$\iiint_E f(x, y, z)\, dV = \iiint_{E_1} f(x, y, z)\, dV + \iiint_{E_2} f(x, y, z)\, dV.$$

Expressing regions conveniently

Problems of this sort often involve good skills in space perception, and as such they can be extremely difficult. Here are two examples.

1. Find an iterated integral expression for evaluating the triple integral of the function $f(x, y, z)$ over the closed region in the first octant bounded by the xy –, yz – and xz – planes and the plane with equation $5x + 3y + z = 15$.

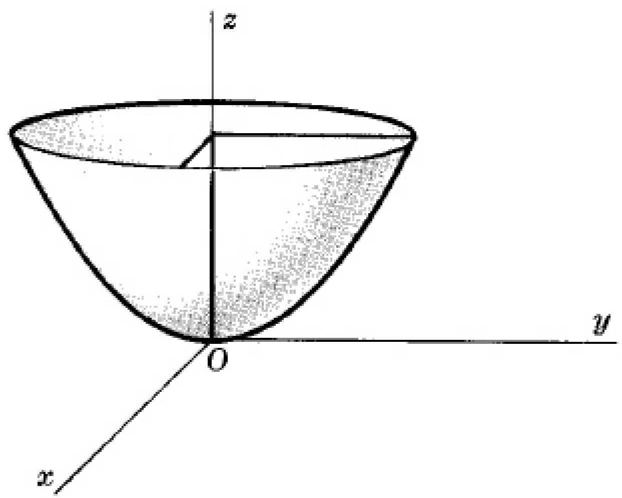
A very rough sketch of the region is given below; it is a triangular pyramid (a ***tetrahedron***) whose base is a right triangle. The vertices of this pyramid are given by the origin, **(3, 0, 0)**, **(0, 5, 0)** and **(0, 0, 15)**.



In this example, x varies between **0** and **3,** while z varies between **0** (the xy – plane) and $15 - 5x - 3y$**,** so the only things left to check are the limits of integration for y expressed in terms of x. These limits are given by the shadow of the region on the xy – plane, and this shadow is the solid triangular region whose vertices are the origin, **(3, 0, 0),** and **(0, 5, 0).** One limit is the line $y = 0$**,** and the other is the line with equation $5x + 3y = 15$**.** If we solve this for y**,** we find that the corresponding limit of integration is the line with the following equation**:**

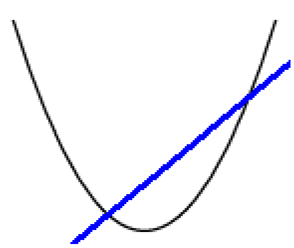
$$y = 5 - \frac{5}{3}x$$

2. Here is an example where the region is bounded by a pair of surfaces. Suppose that the region is bounded by the elliptic paraboloid with equation $z = x^2 + y^2$ and the plane $z = 2x + 3$**.** The first of these surfaces is sketched below.

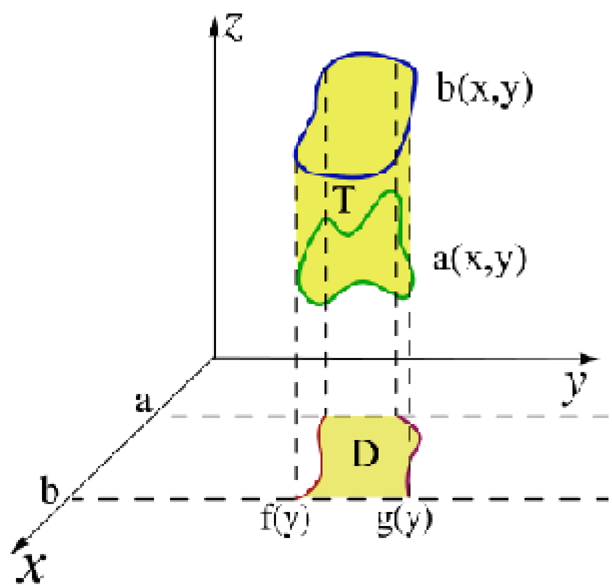


(Source: **http://www.math10.com/en/geometry/analytic-geometry/geometry4/22.jpg**)

One way to obtain some insight into the given region is to look at its intersection with the xz – plane. The intersection of the paraboloid with this plane is just the parabola $z = x^2$**,** while the intersection of xz – plane with the plane $z = 2x + 3$ is merely the line with the same equation**.** This planar cross section is sketched below (but not to scale):



The drawing indicates that the plane $z = 2x + 3$ lies above the elliptic paraboloid, so that the z limits of integration go from $x^2 + y^2$ to $2x + 3$. The corresponding region of integration in the xy – plane is given by the shadow of the original **3 –** dimensional region over the xy – plane; more precisely, these are the feet of perpendiculars dropped from points in the region. Algebraically, this is equal to the set of all points (x, y) such that (x, y, z) lies in the original region. To illustrate this, we shall reprint an earlier illustration**;** observe that **D** is the shadow of **T**.



In order to find the x and y limits of integration over this shadow, we need to find the shadow of the curve at which the two surfaces meet. This shadow curve is given by the equation $x^2 + y^2 = 2x + 3$. This is the equation of some conic in the plane, and it turns out that the latter equation can be rearranged to have the form $(x - 1)^2 + y^2 = 4$, so that the shadow of the intersection curve is a circle. The shadow region in the xy – plane is merely the solid disk bounded by this circle.

It is now fairly straightforward to compute the x and y limits of integration using the disk and boundary circle we have just found. In particular, we can conclude that $-2 \le x - 1 \le 2$, or equivalently $-1 \le x \le 3$. Similarly, the y limits of integration are between $\pm\,\mathbf{sqrt}\left(4 - (x - 1)^2\right)$. Since we have already given the z limits of integration, we are finished.