Comments on Colley, Section 5.5

In single variable calculus, change of variables is an extremely useful technique for rewriting integrals in more computable forms, and not surprisingly there are correspondingly important results for multiple integrals.

Before discussing these, it might be helpful to review some aspects of the single variable theory in a form that will generalize to several variables. The Chain Rule for derivatives and the Fundamental Theorem of Calculus combine to yield the following integral identity; for the sake of simplicity, we assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous on the relevant intervals and that $\boldsymbol{g}$ has a continuous derivative.

$$\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Strictly speaking, when we think of change of variables we think of something that is **reversible:** Not only do we want $\boldsymbol{x}$ to be uniquely expressible in terms of $\boldsymbol{t}$, but ***conversely*** we ***also*** want $\boldsymbol{t}$ to be uniquely expressible in terms of $\boldsymbol{x}$. In order for this to happen, the change of variables function $\boldsymbol{x = g(t)}$ must be either strictly increasing or strictly decreasing.

Suppose first that g is strictly increasing. In this case $\boldsymbol{g(a) < g(b)}$ and the right hand side is just fine. However, if $\boldsymbol{g}$ is strictly decreasing, so that $\boldsymbol{g(a) > g(b)}$**,** then the lower limit in the right hand integral is greater than the upper limit. By convention, this expression is equal to the negative of the integral with the limits written in the usual order. For our purposes it is useful to note that both cases can be combined into a single formula as follows:

Adopt the notation of the preceding formula, and assume that the function $\boldsymbol{g}$ *is either strictly increasing or decreasing with image given by the closed interval* $\boldsymbol{[c, d]}$**.** *Then we have the following change of variables formula:*

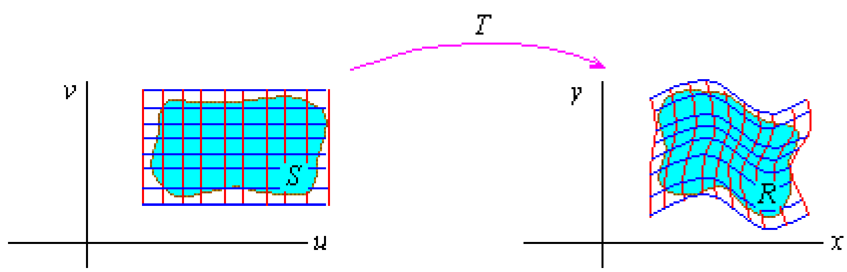
$$\int_c^d f(x)\,dx = \int_a^b f\big(g(t)\big)|g'(t)|\,dt$$

Notice the absolute value signs which surround $\boldsymbol{g'(t)}$ on the right hand side. These play an important role in the generalizations to several variables.

In one dimension, change of variables takes one interval to another; the two intervals often have different lengths, but their shapes are the same. However, in two or three dimensions, both the size and shape can change, so the first step in studying change of variables in multiple dimensions is to become familiar with some of the things that can happen and to adjust the setting so that it reflects these complications.

Transformations of regions

It is useful to view change of variables phenomena in two and three dimensions as geometrical transformations which send the points in one region S into the points in another region R. For the time being we shall restrict attention to the case of two variables.



(Source: http://math.etsu.edu/Multicalc/Chap4/Chap4-4/index.htm)

As indicated in the illustration, the coordinates in the source are denoted by (u, v) and viewed as points in the uv – plane, while the coordinates in the target are denoted by (x, y) and viewed as points in the xy – plane. Analytically, the change of variables transformation is given by a formula like

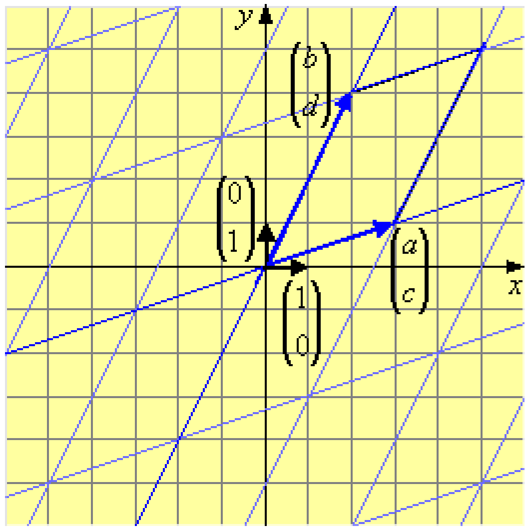
$$T(u, v) = \big(f(u, v),\ g(u, v) \big)$$

Where **(1)** f and g are functions with continuous partial derivatives, **(2)** the system of equations $x = f(u, v)$ and $y = g(u, v)$ can be solved uniquely for u and v either everywhere or "almost everywhere" on R, and **(3)** these solutions are expressible as functions of x and y with continuous derivatives almost everywhere.

Here are some simple but important examples:

<u>Translations.</u> In this case $f(u, v) = u + a$ and $g(u, v) = v + b$ for some fixed vector (a, b). Physically, this corresponds to moving everything a units in the u – direction and b units in the v – direction. The functions f and g obviously have continuous partial derivatives, and clearly we also have unique solutions to the system of equations $x = f(u, v)$, $y = g(u, v)$ given by $u = x - a$ and $v = y - b$. The latter formulas clearly show that u and v have continuous partial derivatives with respect to x and y.

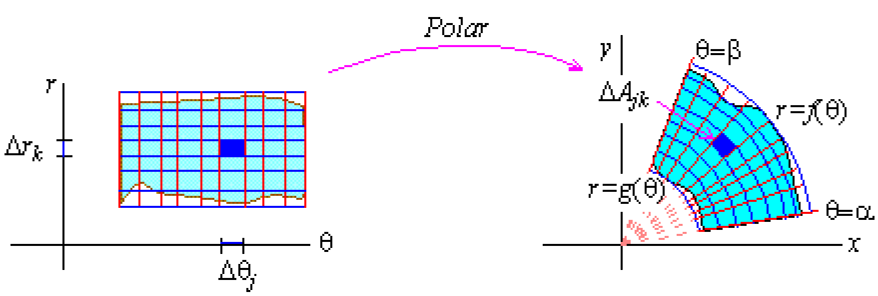
Invertible linear transformations. In this case $x = f(u, v) = au + bv$ and $y = g(u, v) = cu + dv$ for some fixed constants a, b, c, d. As usual, we need to assume that the determinant $ad - bc$ is nonzero in order to solve for u and v.



(Source: www.ies.co.jp/math/java/misc/don_trans/pict.gif)

The solutions for u and v are linear expressions in x and y, so once again these solutions clearly have continuous first partial derivatives.

Polar coordinates. In this case it is customary to view the source as the $r\theta-$ plane, and then the polar coordinate transformation takes the standard form; namely, $x = r\cos\theta$ and $y = r\sin\theta$.



We can solve this uniquely for r and θ in terms of x and y provided r is nonzero and the values of θ lie within some interval whose length is less than 2π; the "almost everywhere" condition means that we can extend everything to the case where r is nonnegative (and possibly zero).

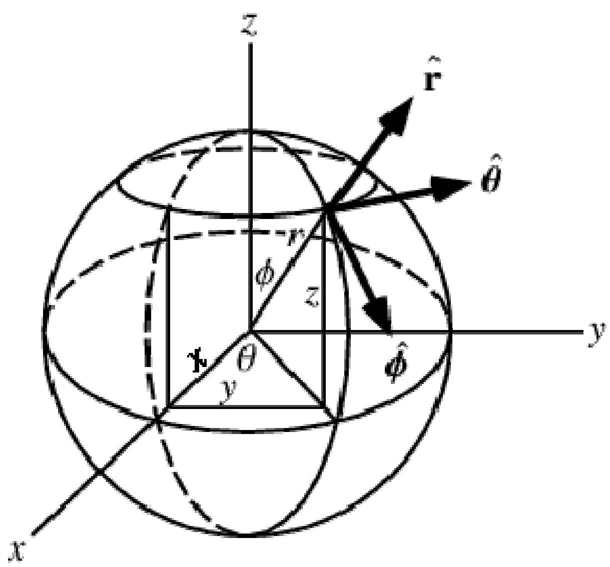
Typical problems

Frequently we are given a change of variables transformation, and the question is to find R given S or vice versa. Sometime the objective is to solve for u and v in terms of x and y or vice versa. Problems of the second type are generally done using algebra. In problems of the first sort, one is usually given the boundary of a region in terms as the set of solutions to some equations. The usual procedure is to solve these equations to find the boundary of the other region in the problem.

Three – dimensional transformations

Everything said thus far carries over to three variables, but in this case one gets a system of three equations in three unknowns. There are similar examples of transformations given by translations and invertible linear transformations, and the polar coordinate transformation has two natural counterparts given by the transformations for cylindrical coordinates ($\boldsymbol{x}$ and $\boldsymbol{y}$ as before in terms of $\boldsymbol{r}$ and $\boldsymbol{\theta}$, together with $\boldsymbol{z}$) and spherical coordinates given by the usual formulas

$$x = \rho\cos\theta\sin\phi,\quad y = \rho\sin\theta\sin\phi,\quad z = \rho\cos\phi.$$



(Source: http://mathworld.wolfram.com/SphericalCoordinates.html)

In this drawing, the variable $\boldsymbol{r}$ corresponds to the variable $\boldsymbol{\rho}$ in the displayed formulas; the three vectors indicate directions in which the coordinate values increase.

The videos at the following site may be helpful for studying the material presented thus far:

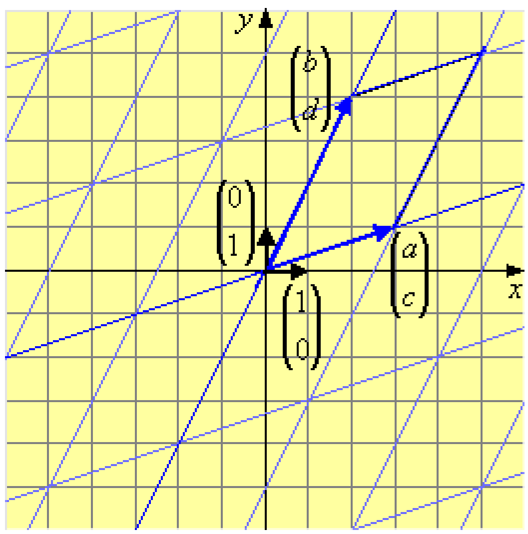
http://gregknese.wordpress.com/2008/05/27/section-148/

Additional material on multivariable calculus by the same author is available at the following site:

http://gregknese.wordpress.com/2008/

The integral formula and important special cases

The multivariable change of variables formula is not all that difficult to state once we understand the concept of (reversible) change of variables as above. Before doing so, it is useful to consider what happens in the particularly simple example of a 2 – dimensional invertible linear transformation as above when the source S is the solid unit square with vertices **(0, 0), (1, 0), (1, 1)** and **(0, 1).** For the sake of convenience, we reproduce the corresponding illustration from earlier.



In this case the target R is a solid parallelogram whose vertices are given by **(0, 0),** (a, c), $(a + b, c + d)$ and (b, d). Considerations from elementary geometry show that the area of R is equal to the ***absolute value*** of the determinant $|ad - bc|$.

In analogy with the single variable change of variables formula, it is natural to expect that the two variable formula says that the integral of a function $f(x, y)$ over R is equal to the integral over S of f, viewed as a function of (u, v), times some correction factor involving the partial derivatives of x and y with respect to u and v. The preceding paragraph suggests that this correction factor might involve determinants; in any case, by that discussion we know this the correction term is a determinant if we are integrating a constant function. This suggestion turns out to be correct.

Specifically, if we are given $x = f(u, v)$ and $y = g(u, v)$ as before, define the ***Jacobian*** of x and y with respect to u and v, to be the function given by the following determinant:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

To save space, we shall often denote the Jacobian by $J(u, v)$. Geometrically, the absolute value of the Jacobian measures the factor by which the transformation T distorts areas near the point at which it is evaluated; this is merely a generalization of our previous observation relating areas to determinants. Therefore one natural guess is that ***the correction factor for change of variables is the absolute value of the Jacobian***, and in fact the **Change of Variables Formula** confirms this:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(u,v)\cdot|J(u,v)|\,du\,dv$$

We shall now check what this means for the previously described examples. If T is a translation, then the Jacobian is equal to **1** everywhere. On the other hand, if T is the previously described linear transformation, then the Jacobian is equal to the absolute value of the determinant $|ad - bc|$. Finally, if T is the polar coordinate transformation, then direct computation shows that the Jacobian is equal to r (the reader should check this!).

Important special case. Suppose that R is the region in the xy – plane bounded by a curve $r(\theta)$ and the rays $\theta = a$ and $\theta = b$ (as in the previous picture, with the lower limit of integration with respect to r set equal to **0**). Then the corresponding region S in the $r\,\theta$ – plane is the solid rectangular region bounded by the lines $\theta = a$, $\theta = b$, $r = 0$, and $r = r(\theta)$, so that we have the following identity:

$$\iint_R f(r,\theta)\,dA = \int_a^b \int_0^{r(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

Three dimensions. There are no surprises, but the notation is more complicated. In this case the Jacobian for the change of variables transformation will be given by a 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

If we once again abbreviate this Jacobian to $J(x, y, z)$, then we obtain the following **3 – dimensional Change of Variables Formula:**

$$\iiint_R f(x,y,z)\, dx\, dy\, dz$$

$$= \iiint_S f(u,v,w) \cdot |J(u,v,w)|\, du\, dv\, dw$$

The special cases of translations and invertible linear transformations are handled exactly as in the 2 – dimensional case, and likewise the special cases of the cylindrical and spherical coordinate transformations are important enough to be described more explicitly. For cylindrical coordinates, this is particularly simple because the Jacobian of (x, y, z) with respect to (r, θ, z) is again equal to r. For spherical coordinates, direct computation shows that the Jacobian of (x, y, z) with respect to (ρ, θ, ϕ) is equal to $-\rho^2 \sin\phi$ (note the sign!), and thus it follows that the correction factor in this case is equal to $\rho^2 \sin\phi$, at least if we assume that ϕ lies between 0 and π.

A noteworthy application of polar coordinates

Note. None of this material will be used subsequently or covered in course examinations. It is merely an illustration of how change of variables can be useful.

Many students in first year calculus courses try to evaluate the indefinite integrals of functions like $\exp(cx^2)$, where c is a constant, and soon realize that the methods and formulas of a first year calculus course do not yield a nice formula for this antiderivative. In fact, as noted in the following document, it is possible to prove rigorously that no such formula exists:

http://math.ucr.edu/~res/math10B/nonelementary_integrals.pdf

If the constant c is negative, then standard convergence tests from single variable calculus (Mathematics 9C) show that the improper integral of the function $\exp(cx^2)$ from $-\infty$ to $+\infty$ converges to a finite value, and it turns out that one can use change of variables to evaluate this improper integral explicitly if $c = -1$. The derivation below is taken from the following source:

http://mathworld.wolfram.com/GaussianIntegral.html

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)} \qquad (1)$$

$$= \sqrt{\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)} \qquad (2)$$

$$= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx}\,. \qquad (3)$$

Here we use the fact that the variable in the integral is a dummy variable that is integrates out in the end and hence can be renamed from x to y. If we make a change of variables to polar coordinates we now see that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\, r\, dr\, d\theta} \qquad (4)$$

$$= \sqrt{2\pi\left[-\tfrac{1}{2}\, e^{-r^2}\right]_0^{\infty}} \qquad (5)$$

$$= \sqrt{\pi}\,. \qquad (6)$$

The function $\exp(-x^2)$ has a bell – shaped graph, and it plays a central role in probability theory (see the previously cited file in the course directory). The preceding formula reflects a curious fact**:** The number π from elementary geometry plays a very significant role in the mathematical theory of probability.