Comments on Colley, Section 5.6

Multiple integrals have numerous applications to questions in the sciences and engineering, and accordingly many scientists, engineers and educators strongly favor the inclusion of something about these applications in a multivariable calculus course. On the other hand, treatments of applications can easily be overdone, and there is always a good chance that such discussions — which involve mathematics, some other subject, and adjustments needed to pass from one subject to the other — will be more confusing than enlightening. Therefore it seems best to stick with a few simple applications that involve only a little physics.

Perhaps the simplest example involves the mass of a uniformly thin object which can be modeled by a closed region in the plane (such an object is called a ***lamina*** in the text). Solid squares, solid triangles, and solid disks are obvious examples, but there are also many others like the regions between two graphs that we have worked with extensively. As in the discussion of triple integrals, if the density of the object is uniform, then the mass is equal to the area times the density, while if the density varies from point to point and is given by a function $\boldsymbol{f(x, y)}$, then the ***mass is equal to the integral of the density*** over the region under consideration. It is important to note that in real life one can expect to find density functions that are not necessarily continuous; for example, part of the object in question may be mainly one substance and part might be mainly another. However, we generally want density functions that are continuous "almost everywhere."

Every reasonable physical object has a ***center of mass*** (or ***center of gravity***) such that in many respects the given object behaves like a point mass concentrated at this center. Integral calculus gives powerful formulas for finding the coordinates of the center of mass. In the formulas below, the density function is called $\boldsymbol{\rho}$.

The ***center of mass*** *of the object is the point* $(\bar{x}, \bar{y})$ *with coordinates*

$$\bar{x} = \frac{M_y}{m} = \frac{\displaystyle\iint_D x\rho(x, y)\, dA}{\displaystyle\iint_D \rho(x, y)\, dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\displaystyle\iint_D y\rho(x, y)\, dA}{\displaystyle\iint_D \rho(x, y)\, dA}.$$

There is a discussion to motivate this formula in the text. Another discussion with illustrations is given in the following document:

http://math.ucr.edu/~res/math10B/centroids.pdf

There is also an analogous set of formulas in three dimensions.

If an object in space fills a region E and has continuous density $\rho(x, y, z)$, its ***moments*** *about the coordinate planes are*

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

$$M_{xz} = \iiint_E y\rho(x, y, z)\, dV.$$

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV.$$

The ***center of mass*** *of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, where m is mass and*

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

In fact, the formula even works if we have a density which is continuous "almost everywhere." As noted before, such objects arise frequently in nature.

One can use these integral formulas together with the computational techniques of preceding sections to compute the centers of mass for many physical objects.

NOTE. ***Moments of inertia will not be covered in this course.***