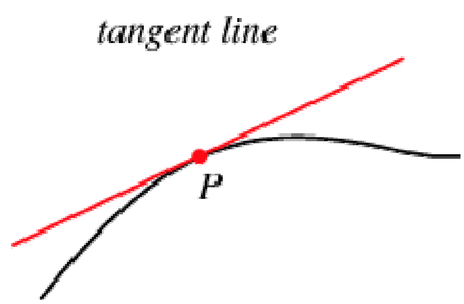
Comments on Colley, Section 6.1

We now know how to integrate functions over intervals in the real line, "good" regions in the coordinate plane, and "good" regions in coordinate **3 –** space. There are also many other sorts of geometric objects over which integrals are defined, and such integrals have a wide range of applications in the sciences and engineering. The simplest examples of this sort are **1 –** dimensional integrals over curves in the coordinate plane or coordinate **3 –** space. These objects are called ***line integrals*.** Some of the material below is taken from the following site:

http://www.math.wisc.edu/~keisler/chapter_13.pdf

The first step is to specify the sorts of curves we shall consider; namely, the ***piecewise smooth curves.*** A vector – valued parametrized curve $\gamma(t)$ defined on an interval $[a, b]$ is said to be ***smooth*** if its coordinate functions $g_1(t)$, $g_2(t)$, ... are continuous and have continuous derivatives on $[a, b]$. The tangent vector to γ at a point c on $[a, b]$ is the vector $\gamma'(t)$ whose coordinates are the derivatives of the coordinate functions $g_1'(t)$, $g_2'(t)$, ... Usually we also want our piecewise smooth curves to be ***regular*** in the sense that the vector derivatives over the separate intervals are never equal to the zero; equivalently, we want the derivative of at least one coordinate function to be nonzero. In such cases, the ***tangent line*** through $\gamma(c)$ is the unique line containing $\gamma(c)$ and $\gamma(c) + \gamma'(c)$.

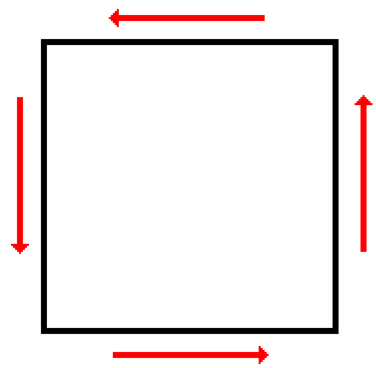


(Source: http://mathworld.wolfram.com/TangentLine.html)

More generally, a piecewise smooth curve is one for which there is a partition of the interval $[a, b]$ given by a finite sequence of intermediate points

$$a = t_0 < t_1 < \ldots < t_m = b$$

such that the restriction of γ to each subinterval $[t_{i-1}, t_i]$ is a smooth curve. The simplest nonsmooth examples of piecewise smooth curves are given by broken lines. There are many situations in the sciences where it is necessary or useful to consider such curves. One basic example of this sort is the unit square with vertices **(0, 0), (1, 0), (1, 1)** and **(1, 1),** with a counterclockwise parametrization.



Here is a sketch of another typical example.

Note that one can define the derivative and tangent line for the parametrized curve at all points except perhaps the partition points t_i. Usually we also want our piecewise smooth curves to be ***regular*** in the sense that the vector derivatives over the separate intervals are never equal to the zero; equivalently, we want the derivative of at least one coordinate function to be nonzero.

As noted in the text, there are two types of line integrals, one for scalar valued functions defined near a curve and another for vector valued functions. For many purposes it is necessary to know that the value of a line integral does not change if we take two parametrizations of the same curve; a verification of this fact is given in Theorems 6.1.4 and 6.1.5 on pages 370 – 372 of the course text. The second type of integral is the more important one for scientific applications, so we shall say a few words about the standard notation for line integrals of vector fields.

DEFINITION

Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

*be a continuous vector valued function on an open rectangle D and let C be a smooth curve in D. The **line integral** of* **F** ***along*** *C,*

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_C P\,dx + Q\,dy,$$

is defined as the definite integral

$$\int_0^L \left(P\frac{dx}{ds} + Q\frac{dy}{ds}\right) ds.$$

Notice that the inner product of $\mathbf{F}$ and $d\mathbf{S}$ is

$$\mathbf{F} \cdot d\mathbf{S} = (P\mathbf{i} + Q\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = P\,dx + Q\,dy.$$

This is why we use both notations $\int_C \mathbf{F} \cdot d\mathbf{S}$ and $\int_C P\,dx + Q\,dy$ for the line integral.

In fact, the second version is standard in most mathematics books, and in fact it is used repeatedly without explanation in the course text.

As noted in Section 6.2, if a curve is closed, which means that $\boldsymbol{\gamma}(\boldsymbol{b}) = \boldsymbol{\gamma}(\boldsymbol{a})$, then one often uses or sees symbolism like the following:

$$\oint$$

Actually, this specific symbol denotes integration with a counterclockwise parametrization; usually the little arrow on the circle does not appear and the notation does not suggest a particular sense (clockwise or counterclockwise) to the curve's parametrization.

The final portion of Section 6.1 deals with numerical methods for computing line integrals. In many practical situations these are indispensable, but this material will not be covered in the course.

Finally, here are solutions to some problems from the lectures.

3−*dimensional line integrals.* Evaluate the line integral

$$\int_\Gamma y\,dx \;+\; z\,dy \;+\; x\,dz$$

when Γ is the curve with parametrization $\gamma(t) = (t, t^2, t^3)$, where $0 \le t \le 1$.

SOLUTION. The standard formulas for line integrals show that the line integral in this problem is given by the following ordinary integral:

$$\int_0^1 t^2\,dt \;+\; t^3(2t\,dt) \;+\; t(3t^2\,dt) \;=\; \int_0^1 (t^2 + 2t^4 + 3t^3)\,dt \;=$$

$$\frac{t^3}{3} + \frac{2t^4}{5} + \frac{3t^4}{4}\Big|_0^1 = \frac{1}{3} + \frac{2}{5} + \frac{3}{4} = \frac{89}{60}$$

Line integrals over closed curves. Evaluate the line integral

$$\int_\Gamma -yx^2\,dx \;+\; x^3\,dy$$

when Γ is the unit circle $\gamma(t) \;=\; (\cos t, \sin t)$, where $0 \le t \le 2\pi$

SOLUTION. In this case the line integral is given by the following ordinary integral:

$$\int_0^{2\pi} -\cos^2 t \sin t(-\sin t\,dt) \;+\; \cos^3 t(\cos t\,dt) \;=\; \int_0^{2\pi} \left(\cos^2 t \sin^2 t \;+\; \cos^4 t\right) dt \;=$$

$$\int_0^{2\pi} \cos^2 t\,dt$$

An antiderivative of $\cos^2 t$ is given by

$$\frac{1}{2} \;+\; \frac{1}{4}\,\sin 2t$$

(this follows from the standard trigonometric identity relating $\cos 2t$ and $\cos^2 t$), and therefore the integal in question is equal to π.