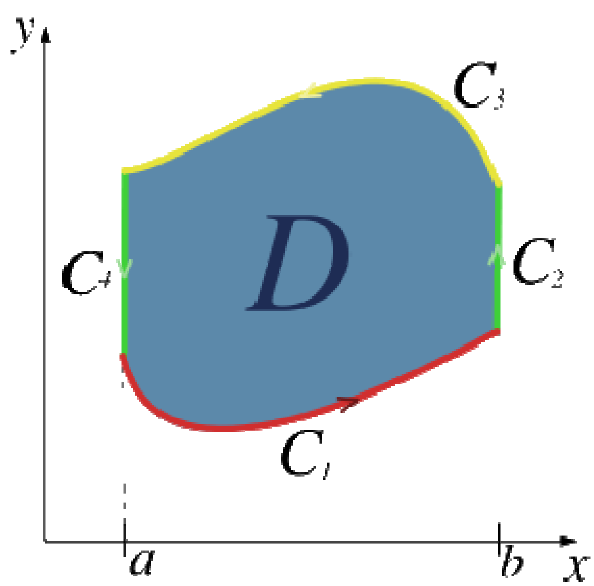
Comments on Colley, Section 6.2

The Fundamental Theorem of Calculus states that the integral of a function over a closed interval $[a, b]$ is equal to the difference of its values at b and a. If we agree that a 0 – dimensional integral is a sum of values of a function at finitely many points with weight factors ± 1, then we can say that a 1 – dimensional integral over an interval is equal to a 0 – dimensional integral over the boundary points. Green's Theorem is an analog which relates the double integral over a region D to a suitably weighted line integral over the boundary ∂D (for the time being, assume that the boundary consists of a single curve Γ). As before, we use some drawings and displays from the following online document, but the treatment of material is much different here.

http://www.math.wisc.edu/~keisler/chapter_13.pdf

Here is a typical example, in which the region D is the sort over which one often considers double integrals:



(Source: http://en.wikipedia.org/wiki/Green%27s_theorem)

Assume now that we are given a vector field $\mathbf{F}(x, y)$ defined on D such that the first and second coordinates of $\mathbf{F}(x, y)$ are functions $P(x, y)$ and $Q(x, y)$ with continuous partial derivatives. We also assume that we have a counterclockwise parametrization of the boundary curve ∂D, which according to the picture may be viewed as a concatenation (stringing together) of the curves C_1, C_2, C_3 and C_4. In this case, we can state the main result of this section as follows:

GREEN'S THEOREM

Let $P(x, y)$ and $Q(x, y)$ be smooth functions on a region D with a piecewise smooth boundary. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA,$$

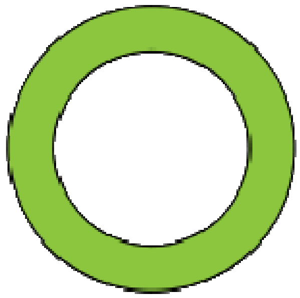
As in the course text, it is fairly straightforward to verify this equation in simple cases like rectangular and triangular regions. One can also use change of variables formulas to establish the result for regions which are transforms of such regions by a mapping of the sort considered in Section 5.5, and it is even possible to prove Green's Theorem in far more general cases by combining such special cases with some additional concepts which are beyond the scope of this course but are alluded to in "Step 2" on page 387 of the course text.

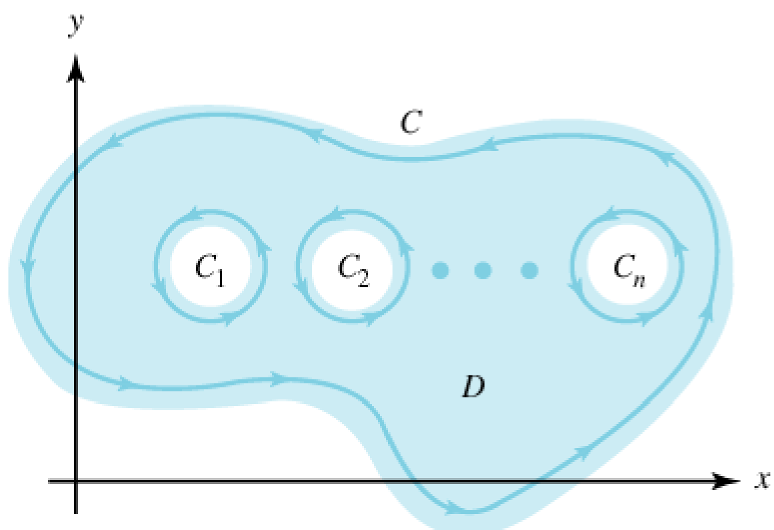
Line integrals and area. One can use line integrals to compute a wide range of quantities. For example, if γ is a curve whose tangent vectors are given by the vector field **F**

$$\gamma'(t) = \mathbf{F}(x(t), y(t))$$

then the line integral of **F** over γ turns out to be the ***arc length*** of the curve. This may not be particularly surprising, but it is probably less apparent that one can also use Green's Theorem to compute the area of a region D using a line integral over the boundary. All we need to do is to find a vector field **F** such that the integrand of the double integral in Green's Theorem is equal to **1.** The standard examples of such integrands are given by $-y\,dx,\quad x\,dy,$ and the average of these two expressions (see Example 2 on pages 382 – 383 of the text). Exercises 6.2.10 and 6.2.13 (on page 389 of the course text) are examples in which area computation by means of line integrals is much easier than using the standard double integral.

Very general forms of Green's Theorem. We have already noted that the theorem holds for fairly general types of regions such that the boundary is a piecewise smooth simple closed curve. However, in some situations one needs a version of Green's Theorem for regions whose boundaries consist of ***several*** closed curves as in the examples illustrated below:





(Sources: http://en.wikipedia.org/wiki/Annulus_(mathematics) , http://math.fullerton.edu/mathews/c2003/cauchygoursat/CauchyGoursatMod/Images/mat0626.gif)

In these cases the double integral over ***D*** is equal to a weighted sum of line integrals over the boundary curves in ∂***D***. The drawings suggest that the boundaries have one ***outer curve*** and one or more ***inner curves***; it is possible to make this precise mathematically, but this requires graduate level mathematics. Given the assertions about inner and outer curves, the appropriate sum of line integrals is equal to the line integral of the outer curve in the counterclockwise sense minus the sum of the line integrals of the inner curves in the same sense (equivalently, plus the sum of the line integrals of the inner curves in the clockwise sense). We shall not use this in exercises or exams, but there are places in mathematics and the sciences where it is useful to know this version of Green's Theorem.

Divergence form of Green's Theorem. An alternate formulation of Green's Theorem is given in Theorem 6.2.3, which is stated, derived, and discussed on pages 384 – 385 of the course text. In Chapter 7 we shall describe and work with **3 –** dimensional analogs of this and the usual versions of Green's Theorem. Since the ∇ (***del*** or ***nabla***) operator plays an important role throughout the rest of this course, for purposes of review we note that the directory

http://math.ucr.edu/~res/math10A

contains files with background information and web links involving the curl and divergence of a vector field as well as the gradient of a scalar valued function.