Comments on Colley, Section 6.3

The starting point is the following observation:

Suppose that Γ *is a curve and the vector field* $\mathbf{F}$ *is the gradient of a function* $\mathbf{g}$. *Then the line integral of* $\mathbf{F}$ *along* Γ *only depends upon the endpoints of* Γ:

$$\int_{\Gamma} P\,dx + Q\,dy = g(\gamma(b)) - g(\gamma(a))$$

This follows quickly from the Chain Rule for partial differentiation and the Fundamental Theorem of Calculus. A derivation is given at the bottom of page 392 in the course text.

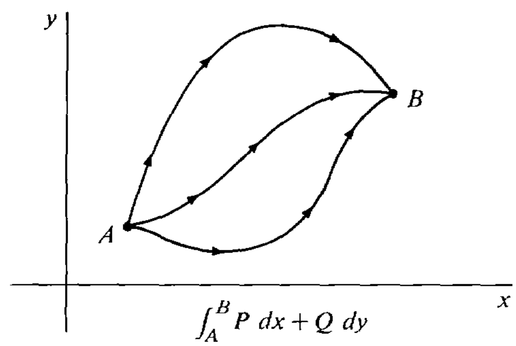

Figure 13.3.1 Independence of path

(Source: http://www.math.wisc.edu/~keisler/chapter_13.pdf)

The integrals along all the paths are equal if (P, Q) is a gradient.

Both physicists and mathematicians were naturally led to consider the following more general question: ***Under what conditions is the line integral of a vector field independent of path?*** Some general observations about this concept appear on pages 391 – 392 of the course text. By the preceding discussion, we know that all gradient vector fields have path independent line integrals.

The equality of mixed second partials yields a simple condition which must hold if a vector field is a gradient:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

In many cases this can be used to show that a vector field is not a gradient. However, even if the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

is satisfied, the vector field is not necessarily a gradient.

EXAMPLE. Suppose that $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$. Then the vector field $\mathbf{F} = (P, Q)$ satisfies the preceding identity. If **F** is a gradient then the line integral over a closed curve will always be zero. However, if we take the line integral of **F** over the unit circle with the usual counterclockwise parametrization $(\cos t, \sin t)$, where t runs from 0 to 2π, then direct calculation shows that the value of this line integral is equal to 2π (verify this!).

The problem in this example is that the vector field is defined on a region which has a "hole" at the origin. It turns out that if a region does not have any "holes" then the condition on partial derivatives is enough to guarantee that the vector field is a gradient (see Theorem 3.5 on pages 395 – 396). In many textbook problems, the vector fields are defined on the entire plane, which satisfies the "no holes" condition.

Here is an example from the course lectures:

Finding potential functions. Find a function g such that $\nabla g(x,y) = (2x+y,\ 2y+x)$.

SOLUTION. First of all, one can check that the vector field **F** in the problem satisfies the criterion to be a gradient. It is defined on the entire plane (a rectangular region), and we also have the identity

$$\frac{\partial}{\partial x}(2y+x) = 1 = \frac{\partial}{\partial y}(2x+y)\ .$$

The first step in finding g is to evaluate the antiderivative

$$\int (2x + y)\, dx = x^2 + xy + h(y)$$

where the constant of integration is some function of y because we are integrating a function of two variables with respect to one of them. To get some information on $h(y)$, we need to differentiate g with respect to y and compare it to the second coordinate of the vector field. When we do this, here is what we obtain:

$$x + 2y = \frac{\partial g}{\partial y} = x + h'(y)$$

From this equation we find that $h'(y) = 2y$, so that $h(y) = y^2 + C$ for some constant C. Therefore the potential function g has the form $g(x,y) = x^2 + xy + y^2 + C$, where C is some undetermined constant. We can solve for C if we are given the value of g at a specific point.

<u>Gradient recognition in three dimensions.</u> In this case the vector field **F** has three coordinate functions that we shall call P. Q and R. If **F** is a gradient, then one obtains the following conditions on the partial derivatives as in the **2 –** dimensional case (using equality of mixed second partial derivatives)**:**

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

These conditions are equivalent to the vanishing of the ***curl*** of **F:** $\nabla \times \mathbf{F} = \mathbf{0}$. Conversely, for a suitable collection of ***simply connected*** regions in **3 –** space, which contains the entire coordinate **3 –** space, and all regions obtained from the latter by removing finitely many points, it turns out that **F** is the gradient of some function g if and only if its curl is equal to zero. The course text includes a few remarks to motivate this statement, but once again material from graduate level courses is needed to formulate and prove everything precisely.

<u>EXAMPLE.</u> If $\mathbf{F}(x, y, z) = (yz, xz, xy)$, then the coordinate functions P, Q, R satisfy the three equations above, so we know that $\mathbf{F} = \nabla g$ for some function g. Since the partial derivative of the function g with respect to x is equal to yz, we know that $g(x, y, z) = xyz + a(y, z)$ for some function a. Since the partial derivative of the latter with respect to y is equal to $xz + a_y$ (where a_y denotes the partial derivative with respect to y) and we know that the partial derivative of g is equal to xz, it follows that $a_y(y, z) = 0$ and hence $a(y, z) = b(z)$ for some function b. Similarly, since the partial derivative of $xyz + b(z)$ with respect to z is equal to $xy + b'$ and we know that the partial derivative of g is equal to xy, it follows that $b'(z) = 0$ and hence $b(z)$ is a constant C. Therefore we have shown that $g(x, y, z) = xyz + C$ for some constant C.