Comments on Colley, Section 7.2

The next step is not particularly surprising. Having defined surfaces, we now want to discuss integration over them. There are obvious analogies with the previous chapter on line integrals, and just as line integrals can be computed using ordinary integrals, in a similar manner we can compute surface integrals using double integrals. However, as before there are complications when we pass from one dimension to two, and some of these are substantial.

As noted in the text, there are two types of surface integrals, one for scalar valued functions defined near a surface and another for vector fields defined near a surface. Likewise, as in the case of line integrals it is often necessary to know that the value of a line integral does not change if we take two parametrizations of the same surface; a partial explanation of this fact is given in Theorems 7.2.4 and 7.2.5 on pages 426 – 428 of the course text.

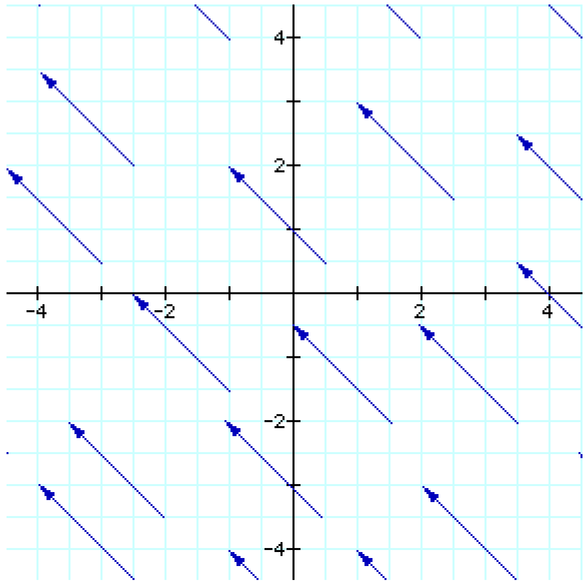
Surface integrals for scalar valued functions are very similar to line integrals for scalar valued functions. In each case, a simple physical model is the problem of computing the mass of an object with variable density (for line integrals the object is a thin wire represented mathematically by a parametrized curve, for surface integrals it is a thin shell represented mathematically by a parametrized surface).

As in the case of line integrals, surface integrals of vector fields are the more important ones for scientific applications. However, the definitions of such surface integrals differ greatly from their line integral counterparts, and in fact they have much different physical interpretations.

Flux integrals

The physical idea behind line integrals of vector fields is based upon the concept of ***energy*** (or ***work***). In contrast, the physical idea behind surface integrals of vector fields is based upon the concept of ***flux*** which we shall now describe.

One of the simplest physical models involves fluid flowing through a porous surface at some given time. We can model the flow of the fluid at that time by a vector field; the value of the vector field at a point will represent the velocity of the fluid flow at that point. For example, if the vector field $\mathbf{F}(x, y)$ has the constant value $(-3/2, 3/2)$, then the fluid flow is in the northwest direction everywhere at a constant rate of speed, which is given by $|\mathbf{F}| = (3/2)\cdot\mathrm{SQRT}(2)$.

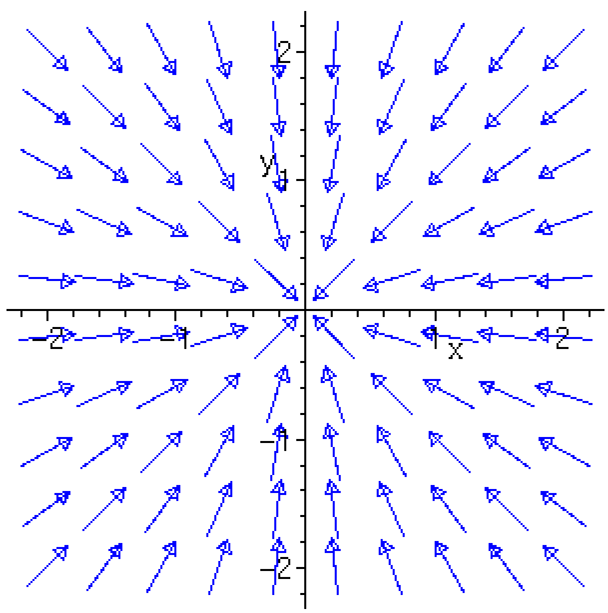


(Source: http://people.math.gatech.edu/~carlen/2507/notes/vectorCalc/vectorfields/vf.gif)

Likewise, if we take the **2 –** dimensional vector field

$$\mathbf{F}(x,y) = \frac{1}{\sqrt{x^2 + y^2}}(-x,-y)$$

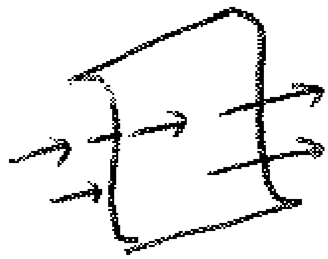
which is defined at all points except the origin, then at each point the fluid is flowing at a unit speed towards the origin in a straight line ("the fluid is going down the drain").



(Source: http://www.math.uic.edu/~math210/newlabs/lab7/images/lab7-11.gif)

In each of these examples the lengths of the vectors are the same everywhere, but certainly one can also give examples where the length is variable. For instance, in the second case one could simply take the vector field $(-x, -y)$; with this change, the speed at a point is equal to the distance between the point and the origin. Obviously, it is also possible to give many similar **3 –** dimensional examples.

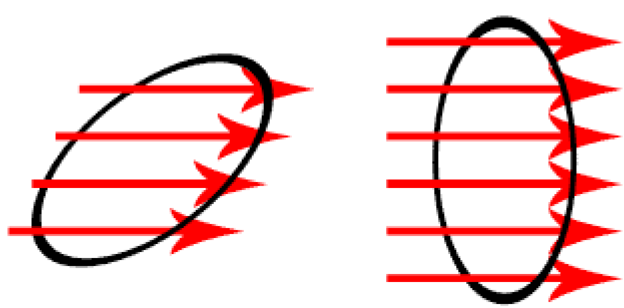
Suppose now there is a porous surface (for the **3 –** dimensional case) or curve (for the **2 –** dimensional case) which lies inside the region where the fluid is situated. One frequently arising issue is to determine the rate at which the fluid is flowing past the surface or curve; this rate is called the ***flux*** of the fluid through the surface or curve.



(Source: http://www.math.utah.edu/~erin/22101/flux.pdf)

In any discussion of flux it is ***absolutely necessary*** to pay some attention to senses and orientations. More precisely, we need to specify a ***preferred sense*** (or orientation) ***for the normal direction*** to the surface or curve in order to have meaningful notions of the fluid moving into and out of the surface or curve. In the preceding illustration, the orientation sense points to the right; another option would be the sense pointing in the exact opposite direction.

As in other discussions of applications, it is best to begin with simple cases. Suppose we start with a flat surface and a fluid for which the velocity at all points is constant (in both speed and direction).



Flux varies by how the boundary faces the direction of flow.

(Source: http://upload.wikimedia.org/wikipedia/commons/7/72/Flux_diagram.png)

As the illustration suggests, the flux through the flat surface depends upon the angle between the surface and the vector field; if the normal vector to the surface points in the same direction as the vector field, then the flux is given by the length of the vector field, if there is some acute angle between these vectors then the flux is some percentage of the length, and if the vector field is perpendicular to the normal vector then nothing is flowing past the surface at that point, so the flux is zero. For obtuse angles one gets a flux which is negative (physically, an inward rather than an outward flow), and if the normal vector points in the opposite direction to the vector field, then the flux is the negative of the length of the vector field. If we examine things more closely, we obtain the following simple but important formula, in which **n** is the preferred normal direction vector for the surface and **v** is the (constant) value of the vector field:

$$\textit{\textbf{Flux over a flat surface}} \;=\; (\mathbf{n}\cdot\mathbf{v})\cdot(\textit{Area of surface})$$

Suppose now that we have a smooth parametrized surface whose parametrization is one – to – one everywhere (***almost everywhere will <u>not</u> suffice!***), and for the sake of convenience assume that the preferred direction of the normal vector is given by $\mathbf{N}(\boldsymbol{u},\boldsymbol{v})$ everywhere (if the domain is connected in the sense that any two points can be joined by a broken line which lies entirely in the domain, then the only other option would be given by the negative of the standard normal vector — in other words, the ***opposite orientation***). Then we can use the previous ideas of splitting the domain into small rectangles, estimating the flux over the surface pieces which correspond to the individual rectangles, adding these together, noticing that the approximations get better if we take smaller rectangles, and observing that the limit of these, which is a surface integral, should be the flux across the oriented surface (at the last step there is the usual issue of making sure that a limit exists, which requires input from more advanced courses). This leads to the following surface integral formula for the flux over a surface ***S***:

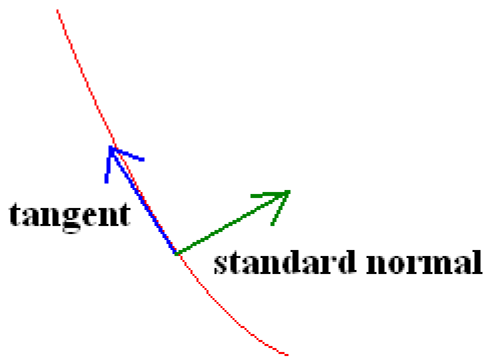
$$\textit{\textbf{Flux}} \;=\; \iint_S (\mathbf{F}\cdot\mathbf{N})\, d\boldsymbol{S}$$

Note that if we choose the opposite orientation, then the value of the flux changes sign. Frequently the flux integral for a vector field over an oriented surface is written (a little) more compactly as follows:

$$\iint_S \overrightarrow{\mathbf{F}}\cdot d\,\overrightarrow{\mathbf{S}}$$

Formally, we may think of $d\mathbf{S}$ as equal to $\mathbf{N}\,dS$. One advantage of this notation is that it is much easier to apply when working specific problems.

<u>Flux integrals for curves.</u> At the beginning of the discussion of flux, we mentioned that there was a **2 –** dimensional version of this concept involving flux through curves rather than surfaces, so we shall now discuss the adaptations needed in order to discuss the planar version of the theory. The main thing we need will be a concept of normal vector to a regular curve. We shall define the standard oriented normal direction to a parametrized curve $\gamma(t) = (x(t), y(t))$ by the formula $\mathbf{N}(t) = (y'(t), -x'(t))$. It follows immediately from the definition that $\mathbf{N}(t)$ is perpendicular to the tangent vector $\gamma'(t)$; we have chosen the sign so that if γ is moving in a counterclockwise direction then $\mathbf{N}$ is pointing away from the origin. For example, if $\gamma(t)$ is the usual counterclockwise circle $(\cos t, \sin t)$, then $\gamma'(t)$ is $(-\sin t, \cos t)$, and $\mathbf{N}(t) = (\cos t, \sin t)$.



If we analyze the flux in this setting as we did in the 3 – dimensional case, it will follow that the flux of a ***<u>planar</u>*** vector field $\mathbf{F} = (P, Q)$ with respect to a curve Γ is equal to the following line integral:

$$\int_{\Gamma} -Q\,dx + P\,dy$$

Equivalently, the flux is the line integral of the scalar function $(\mathbf{F} \cdot \mathbf{N})$ which is the analog of the function which appears in the **3 –** dimensional case.

<u>Computing flux integrals for surfaces.</u> Here is yet another way of writing the flux integral for a vector field $\mathbf{F} = (P, Q, R)$:

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

It is **extremely important** to note the ordering of the differential expressions ***dx*, *dy*,** and ***dz*** in this integrand, for ***if the wrong orderings are used in a computation then the result will almost inevitably be incorrect.*** This expression is very useful because it reflects the following formula for computing a flux integral for the oriented parametrized surface in terms of a double integral over the domain ***D*** on which the parametrization is defined:

$$\iint_D \left(P\,\frac{\partial(y,z)}{\partial(u,v)} + Q\,\frac{\partial(z,x)}{\partial(u,v)} + R\,\frac{\partial(x,y)}{\partial(u,v)}\right) du\,dv$$

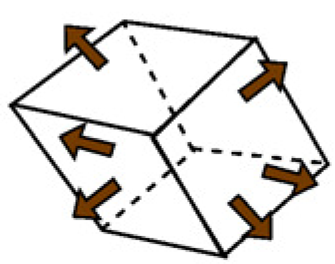
The importance of paying careful attention to the order of differentials is apparent from this formula, for if the ordering of two variables is switched in any of the Jacobians, then the corresponding summands will change sign.

Piecing together orientations. In the preceding discussion we assumed that our surfaces were smoothly parametrized and simply took the standard orientation given by the cross product of the partial derivatives. However, if we have something like a piecewise smooth surface, where each piece has a separate parametrization, then more care is needed in choosing normal directions.

Textbook exercise 5 for Section 7.1 is a good example to consider. This surface was given by the six faces of the boundary for the cube $-2 \leq x, y, z \leq 2$, and we gave the following parametrizations for the six faces of the cube:

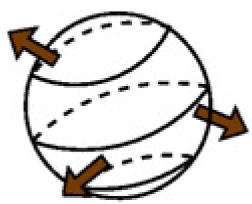
$$(-2, u, v) \quad (2, u, v) \quad (u, -2, v) \quad (u, 2, v) \quad (u, v, -2) \quad (u, v, 2)$$

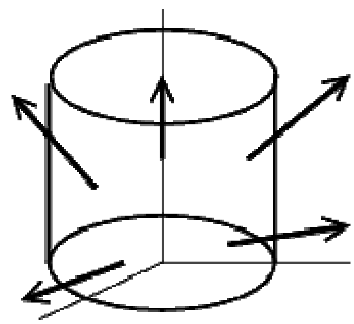
If we take the cross products of the partial derivatives for each parametrization, we find that the standard normal directions point towards the inside of the cube in the first, third and fifth cases, and they point towards the outside in the other three cases. Generally when we are working with something like a cube we want to choose orientations so that ***the normal is always pointed towards the outside***, as in the following illustration



(Source: http://www.math.rutgers.edu/~greenfie/mill_courses/math251/diary4.html)

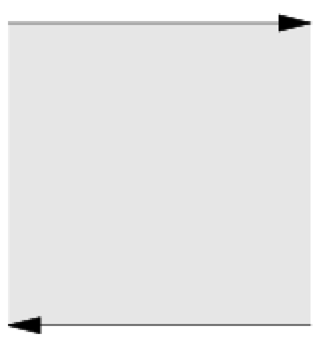
This forces an important modification to the formulas for surface integrals. Instead of simply taking the normal to be the usual $\mathbf{N}(u, v) = \mathbf{X}_{1st}(u, v) \times \mathbf{X}_{2nd}(u, v)$, we need a global system of orientations $\boldsymbol{\Omega}(u, v)$ which somehow can be fit together coherently. In the case of a closed bounded surface like a sphere or cylinder boundary or cube boundary, it is always possible to do this using outward pointing normals, and in such cases we can take $\boldsymbol{\Omega}(u, v)$ over a given piece to be plus or minus the usual $\mathbf{N}(u, v)$. Of course, one must look very carefully at the parametrizations over the individual pieces when choosing the global system of orientations (see the criterion below). The spherical and cylindrical cases are illustrated below.



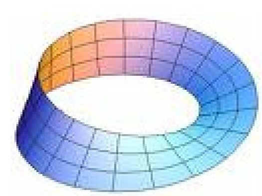


(Sources: http://www.math.rutgers.edu/~greenfie/mill_courses/math251/diary4.html , http://www.math.umn.edu/~nykamp/m2374/readings/surfintex/surfintex13x.png)

On the other hand, if the surface has one or more boundary curves, then it is not always possible to choose reasonably coherent orientation systems. The basic example of this sort is given by the **Möbius strip,** which is formed by taking a strip of paper and gluing two of the ends together as indicated below:



Möbius strip



(Sources: http://mathworld.wolfram.com/MoebiusStrip.html , http://virtualmathmuseum.org/Surface/moebius_strip/moebius_strip.html)

This surface has the curious property that, if you travel along the closed path of the center curve, then at the end you will be upside down from your original position. The following artwork by M. C. Escher illustrates this phenomenon:

(Source: http://richardwiseman.files.wordpress.com/2009/03/escher-mobius_strip_ii.jpg)

Here is an animated (and less creepy!) illustration of the same point:

http://www.videmo.com/yvideo.php?i=TDQ4UkJzcWuRpeFAyQjQ&mathematica-mbius-strip-traversal=

This means that there is a closed smooth path on the surface for which one cannot choose a unique, continuously varying normal direction to the surface, and hence ***there is no reasonable choice for a preferred normal direction*** to the Möbius strip, and consequently ***we cannot discuss flux integrals for such a surface***. Fortunately, such nonorientable surfaces very rarely arise in the applications of vector calculus to physics and engineering, and in particular this never happens for closed surfaces like the ones mentioned previously.

How can we recognize coherent global systems of orientations?

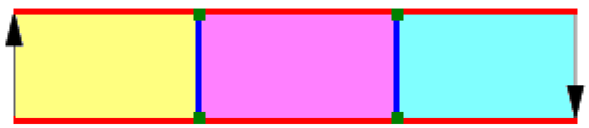
There is no simple answer to this question, but we shall try to explain one key aspect. By definition, a piecewise smooth surface is formed from finitely many parametrized surfaces defined over the sorts of regular domains for which one can compute double integrals using iterated ordinary integrals; in other words, they are defined by inequalities of the form $a \leq u \leq b$ and $g(u) \leq v \leq h(u)$. Suppose we are given an arbitrary orientation system $\mathbf{\Omega}$ over the pieces; then it is possible to choose the parametrizations $\mathbf{X}_j$ such that the orientations obtained from the parametrization always match W, for if one does not, then one can replace it by the new parametriation $\mathbf{X}_j\,(\mathbf{u}, -\mathbf{v})$, which is defined on the set of points satisfying the inequalities $a \leq u \leq b$ and $-h(u) \leq v \leq -g(u)$. By construction, the boundary of each piece defines a simple closed curve $\mathbf{\Gamma}_j$. If a surface has a nonempty boundary, then the boundary of the entire surface is made up of the smooth pieces which come from at least some of these curves $\mathbf{\Gamma}_j$, but usually most of the smooth pieces do not lie on the boundary.

With these conventions, we can state the basic criterion for orientability which is important for vector calculus.

CRITERION. *A system of global orientations is coherent if and only if the sum of the line integrals over the curves* $\mathbf{\Gamma}_j$ *is equal to the sum of the line integrals of the boundary curves for the surface.*

For example, if we take the cube as before, then it has no boundary curves, and in fact if we choose parametrizations as above so that the normals always point out word, then the sum of the line integrals on the boundaries of the separate pieces will add up to zero for this system of orientations. The computation is fairly straightforward but somewhat messy, so we shall not give the details. Similarly, outward normals yield coherent orientation systems for the sphere and cylinder.

On the other hand, suppose we try to do this with the Möbius strip. In the drawing below we have decomposed this surface into three well – behaved smooth pieces on which one has separate one – to – one parametrizations..



There are two ways of orienting each of the three smooth pieces, and these yield a total of 8 global orientation systems. These orientations correspond to choosing the clockwise or counterclockwise senses for the boundary curves of the three pieces.

A coherent choice of orientations would be one for which the sum of these line would reduce to the line integral for the boundary curve, which is represented by the pair of colored red in the drawing. However, none of the 8 global orientation systems have this property; in fact, for each choice the sum turns out to involve 2 times the line integral of at least one vertical curve in the drawing (note that the curves in the left and right are the same, with the senses identified as indicated by the arrows).