Comments on Colley, Section 7.4

Undergraduate physics courses for students in the physical sciences and engineering use vector analysis extensively. In particular, Stokes' Theorem and the Divergence Theorem are important in physics because they play key roles in the derivation of many basic laws in the subject. This section of the text discusses some mathematical issues related to such uses of material from multivariable calculus. In this commentary we shall discuss some of these topics from a slightly different viewpoint.

UNIFIED NOTATION FOR INTEGRALS. Frequently we shall discuss results on integrals that are valid for ordinary, double and triple integrals. Eventually it becomes awkward and tedious to formulate separate statements for each of these dimensions. One further disadvantage of this is that one can set up analogous theories of multiple integration for functions of 4, 5, or n variables (where n must be a positive integer; setting things up for infinitely many variables is more problematic), and obviously it is at best clumsy and at worst impossible to express such integrals using repeated integral signs. In order to streamline the notation and emphasize the similarities between the results in various dimensions, we shall often use notation like

$$\int_D f(\vec{\mathbf{x}})\, d\mu$$

to denote the n – tuple integral of a real or vector valued function f defined on some region of coordinate n – space. Although in many cases everything goes through for all n, for the purposes of this course nothing will be lost by assuming that n is one of 1, 2, 3.

FACTS ABOUT CONTINUOUS FUNCTIONS. Before proceeding, we shall mention some general properties of continuous functions that are fundamentally important but often do not receive much attention in calculus courses. Given a point $\mathbf{p}$ in coordinate n – space and a positive real number r, as usual we shall define the closed disk $D_r(\mathbf{p})$ with radius r and center $\mathbf{p}$ to be the set of all points $\mathbf{y}$ in coordinate n – space such that $|\mathbf{y} - \mathbf{p}| \leq r$. We then have the following result:

If f is a continuous real valued function on $D_r(\mathbf{p})$, then $D_r(\mathbf{p})$ takes a maximum value M and a minimum value m at some point(s) of $D_r(\mathbf{p})$. Furthermore, if z is an arbitrary real number between m and M, then there is some point $\mathbf{w}$ in $D_r(\mathbf{p})$ such that $f(\mathbf{w}) = z$.

In fact, this result remains valid if one replaces the closed disk with many other types of closed regions (for example, the sorts of regions in the coordinate plane or **3 –** space over which double and triple integrals can be evaluated in terms of iterated integrals), but the statement above is all that we shall need.

Differentiation under the integral sign

To illustrate the unified notational convention for ordinary and multiple integrals, we shall state a basic result on the topic in the heading for this subsection.

THEOREM. *Suppose that $f(\mathbf{x}, t)$ is a continuous function which is defined for all $(\mathbf{x}, \mathbf{t})$ such that $\mathbf{x}$ lies in some n – dimensional region D and t lies in some interval J, and assume further that f has a continuous partial derivative with respect to t. If*

$$g(t) = \int_D f(\vec{\mathbf{x}}, t)\, d\mu$$

then g is continuously differentiable on the interval J, and its derivative is given by

$$g'(t) = \int_D \frac{\partial f}{\partial t}(\vec{\mathbf{x}}, t)\, d\mu\ .$$

We should note another aspect of our unified notation here; namely, the formula works for both scalar and vector valued functions (recall that continuity and differentiability are definable for vectors in terms of coordinates). The **1 –** dimensional case of this result is Exercise 22 on page 360 of the text. Further information appears in the following online documents:

http://en.wikipedia.org/wiki/Differentiation_under_the_integral_sign

http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html

Deriving consequences of Stokes' Theorem and the Divergence Theorem

At the beginning of Section 7.4 in the course text there are several formulas called ***Green's formulas*** (also known as ***Green's identities***) which are derived from the Divergence Theorem, and some consequences of such formulas are discussed. Here are a few more direct consequences of Stokes' Theorem and the Divergence Theorem which appear in physics and engineering.
:

$$\iiint_V (\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}))\, dV = \oiint_S \mathbf{F} \times \mathbf{G} \cdot d\mathbf{S}.$$

One can prove this by applying the Divergence Theorem to the cross product of the vector fields $\mathbf{F}$ and $\mathbf{G}$. Details are left to the reader. Similarly, one can apply Stokes' Theorem to a cross product of two vector fields and obtain the following:

$$\int_\Sigma (\mathbf{F}\,(\nabla \cdot \mathbf{G}) - \mathbf{G}\,(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\,\mathbf{F} - (\mathbf{F} \cdot \nabla)\,\mathbf{G}) \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} (\mathbf{F} \times \mathbf{G}) \cdot d\mathbf{r}.$$

Applications to the laws of physics

Instead of continuing with the mathematical discussion at this point, we shall consider a specific example from physics; namely, the ***Gauss Law for electrostatic fields***. The basic idea is that one has an electrostatic force field $\mathbf{E}$, which is a 3 – dimensional vector field over some region in space, and a closed surface S which bounds some subregion D. One can state the integral version of Gauss' Law for electrostatic fields as

$$\iint_S \mathbf{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

where q is the total charge inside D and ε_0 is some fundamental physical (permittivity) constant. We shall now explain how one can derive a fundamentally important alternate version of this law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

in which ρ denotes the electrical charge density at a given point. This derivation is entirely formal and requires little or no understanding of the underlying physics. About the only physical input needed is that the total charge q inside D is equal to the integral of the charge density ρ over D. If we combine this fact with the integral form of Gauss's Law and the Divergence Theorem, we obtain the following equation:

$$\iiint_D (\nabla \cdot \mathbf{E})\, dV = \frac{1}{\varepsilon_0} \iiint_D \rho\, dV$$

This looks very much like the second equation above; the main difference is that the new equation states that the integrals of $\nabla \cdot \mathbf{E}$ and ρ / ε_0 over D are equal, where there are many different choices for D. In order to derive the second equation, we need the following sort of ***retrieval principle***:

> *Suppose that* g *and* h *are continuous functions on a region such that for all closed disks* D *in the region, the integrals of* g *and* h *over* D *are equal. Then* $g = h$.

A proof of this result is given in the file

http://math.ucr.edu/~res/math10B/multidifferentiation.pdf

in the course directory.

Similar discussions for the other three fundamental laws of electromagnetism are carried out on pages 461 – 465 of the course text; the previously cited document in the course directory proves a retrieval principle for surface integrals which is analogous to the one we have stated for functions. Specifically, for such a result we start with two vector fields $\mathbf{F}$ and $\mathbf{G}$ whose flux integrals over a suitably large class of flat 2 – dimensional disks are equal, and hence the conclusion is that $\mathbf{F} = \mathbf{G}$.

Related methods yield a derivation for the ***heat conduction equation***, which is discussed in Exercisses 6 – 10 on pages 467 – 468 of the course text and also in the online document **http://math.ucr.edu/~res/math10B/heat95.pdf**. For the sake of completeness, here are some references dealing with the derivation of wave motion equations; in this context also see Exercise 15 on pages 468 – 469 of the course text.

http://www.mathphysics.com/pde/index.html

http://www.mathphysics.com/pde/WEderiv.html

http://www.mathphysics.com/pde/Maxwell.html

ADDENDUM: *Mean value theorems for integrals*

This is a further note related to the Differentiation Theorems in the online document **http://math.ucr.edu/~res/math10B/multidifferentiation.pdf.** Frequently this result is derived using a so – called Mean Value Theorem for integrals. Versions of this result are stated at a couple of points in the course text; we shall state the result here and explain how one can derive the Differentiation Theorems from it.

Section 5.6 of the course defines the ***average value*** for a continuous real valued function f on a reasonable region D by the formula

$$\overline{f_D} = \frac{1}{\mu(D)} \int_D f(\vec{x})\, d\mu$$

where the ***measure*** of D, written $\mu(D)$**,** is the length, area or volume of D depending upon whether the dimension n is **1, 2,** or **3**. For the sake of completeness, we should note that similar definitions also exist in higher dimensions.

One version of the Mean Value Theorem from single variable calculus states that, in the **1 –** dimensional case, the average value of the function is in fact equal to the function at some point of the region D (which in this case is an interval) over which one is integrating. Important generalizations of this to 2 and 3 variables are mentioned very informally on pages 473 and 445 of the course text. Our purpose here is to give a formal statement of these results when D is a closed disk of the form $D_r(\mathbf{p})$.

Integral Mean Value Theorem. *Suppose that* f *is a continuous real valued function on* $D_r(\mathbf{p})$ *as above. Then there is some point* Y *in* $D_r(\mathbf{p})$ *such that*

$$\overline{f_D} = f(Y).$$

Derivation of the Integral Mean Value Theorem. We know that f assumes maximum and minimum values M and m on $D_r(\mathbf{p})$, so that $m \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x}$ in $D = D_r(\mathbf{p})$. Basic properties of integrals (as in the commentaries for Chapter 5) then yield the integral inequalities

$$\int_D m \, d\mu \leq \int_D f(\vec{\mathbf{x}}) \, d\mu \leq \int_D M \, d\mu$$

and since the integral of a constant function k over $\mathbf{D}$ is equal to $k\,\mu(D)$, it follows immediately that we may rewrite these inequalities as

$$m \leq \frac{1}{\mu(D)} \int_D f(\vec{\mathbf{x}}) \, d\mu \leq M$$

and by the previously stated intermediate value theorem for continuous functions on $D_r(\mathbf{p})$ it follows that the term in the middle is equal to $f(Y)$ for some Y in the disk $D_r(\mathbf{p})$.

DERIVATION OF THE DIFFERENTIATION THEOREMS. Here is the derivation based upon the Mean Value Theorem(s). Throughout this discussion the point $\mathbf{p}$ in the domain D will remain fixed, and to simplify notation we shall denote $D_r(\mathbf{p})$ by rD. Throughout this discussion we shall assume that r is always so small that rD is contained in the region on which the function f is defined.

Let $r > 0$ be as above. By the Mean Value Theorem, there is some point Y_r in the disk rD such that

$$f(Y_r) = \frac{1}{\mu(rD)} \int_{rD} f(\vec{\mathbf{x}}) \, d\mu$$

and if we take the limit of the left hand side as $r \to 0$ we merely obtain $f(\mathbf{p})$. Of course, this means that the limit of the right hand side as $r \to 0$ and thus the limit of the latter is also equal to $f(\mathbf{p})$. This is precisely the statement of the Differentiation Theorem.