An example involving the Divergence Theorem

PROBLEM. Let S be the boundary surface for the the region D determined by the xy-plane, the cylinder $x^2 + y^2 = 4$, and the plane $x + z = 6$ (there is a drawing of this on page 10 of the file `comments0703.pdf` in the course directory). Find the flux of the vector field $\mathbf{F}(x, y, z) = (x^2 + \sin z, xy + \cos z, e^y)$ over the surface S, where the latter is oriented with respect to the outward pointing orientation.

Computing the surface integral directly would require evaluating at least three separate double integrals, corresponding to the top, bottom and lateral faces of S. The Divergence Theorem reduces the problem to evaluating a single triple integral.

Solution. The first step is to compute the divergence of $\mathbf{F}$, and standard differentiation rules show that $\nabla \cdot \mathbf{F}(x, y, z) = 2x + x = 3x$.

The easiest way to evaluate the triple integral is to use cylindrical coordinates, and with respect to this choice the region D is defined by the inequalities $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 6 - x = 6 - r\cos\theta$. Therefore the Divergence Theorem implies that the flux is equal to

$$\int_0^{2\pi} \int_0^2 \int_0^{6 - r\cos\theta} 3r\cos\theta \cdot r \, dz \, dr \, d\theta \ .$$

If we perform the iterated integrations from the inside out, we find that this triple integral is equal to

$$\int_0^{2\pi} \int_0^2 (18r^2 \cos\theta - 3r^3 \cos^2\theta) \, dr \, d\theta \quad = \quad \int_0^{2\pi} (48\cos\theta - 12\cos^2\theta) \, d\theta \quad =$$

$$48\sin\theta - 6\left(\theta + \tfrac{1}{2}\sin 2\pi\right) \Big|_0^{2\pi} \quad = \quad -12\pi \ .$$