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Mathematics 10B–010, Fall 2009, Examination 1

Answer Key

Note. There are drawings for problems 2–4 in the file

`exam1f09drawings.pdf`

(which is in the course directory).

1. [*20 points*] Evaluate the integral of $8yz^3 \sin x$ over the rectangular solid $0 \le x \le \pi$, $1 \le y \le 2$, and $0 \le z \le 1$.

SOLUTION

The triple integral is equal to the following iterated integral (one could also set this up using any reordering of the three variables):

$$\int_0^\pi \int_1^2 \int_0^1 8yz^3 \sin x \, dz \, dy \, dx$$

At the first step we integrate with respect to z and get

$$\int_0^\pi \int_1^2 2yz^4 \sin x \Big|_{z=0}^{z=1} dz \, dy \, dx \quad = \quad \int_0^\pi \int_1^2 2y \sin x \, dy \, dx$$

while at the second step integrate with respect to y and get

$$\int_0^\pi y^2 \sin x \Big|_{y=1}^{y=2} dy \, dx \quad = \quad \int_0^\pi 3 \sin x \, dx \ .$$

The latter is equal to

$$-3 \cos x \Big|_{x=0}^{x=\pi} \quad = \quad 6 \ .$$

2. [*25 points*] Express the integral of $f(x,y) = y$ over the region bounded by the curves $y = x^2 - 1$ and $y = 1 - x^4$ as an iterated integral $\int\int\ y\,dy\,dx$. You need not evaluate the iterated integral. [*Hint:* The curves meet at two points on the x-axis.]

SOLUTION

A sketch shows that the curves cross at $x = \pm 1$ on the x-axis, and between these values of x the graph of $x^2 - 1$ lies below the graph of $1 - x^4$. Therefore the double integral of y over this region is equal to the following iterated integral:

$$\int_{-1}^{1} \int_{x^2-1}^{1-x^4} y\,dy\,dx$$

Although the problem does not ask for it, the value of the iterated integral is equal to

$$\int_{-1}^{1} \left.\frac{y^2}{2}\right|_{y=x^2-1}^{y=1-x^4} dx \quad =$$

$$\frac{1}{2} \cdot \int_{-1}^{1} (1-x^4)^2 - (x^2-1)^2\,dx \quad = \quad \frac{1}{2} \cdot \int_{-1}^{1} x^8 - x^4 - 2x^2\,dx\ .$$

Since the integrand is an even function (powers of x^2), is equal to twice the integral of the given function from 0 to 1, and therefore the last integral equals

$$\int_0^1 x^8 - x^4 - 2x\,dx \quad = \quad \frac{x^9}{9} - \frac{x^5}{5} - \left.\frac{2x^3}{3}\right|_0^1 \quad = \quad \frac{26}{45}\ .$$

See the last page for still further calculations related to this problem.

3. [*25 points*] Interchange the order of integration in the following expression:

$$\int_1^2 \int_0^{2-x} f(x,y)\, dy\, dx$$

SOLUTION

A sketch of the region shows that it is a solid triangular region with vertices at $(1,0)$, $(2,0)$ and $(1,1)$ with boundary curves $y = 0$, $y = 1$, $x = 1$ and $y = 2 - x$. The last curve lies to the right of the vertical line $x = 1$, and if we solve $y = 2 - x$ for x we obtain $x = 2 - y$. Therefore the region is describable by $0 \le y \le 1$ and $1 \le x \le 2 - y$, so that the iterated integral is also equal to

$$\int_0^1 \int_1^{2-y} f(x,y)\, dx\, dy \ .$$

4. [*30 points*] Use polar coordinates to express the double integral

$$\int\int_R x\,dx\,dy$$

as a product of two ordinary integrals, where R is the pie-shaped region whose polar coordinates satisfy $0 \leq r \leq 1$ and $-\alpha \leq \theta \leq \alpha$ for some α between 0 and $\pi/2$. You need not evaluate the ordinary integrals.

SOLUTION

By the change of variables formula, the double integral equals

$$\int_{-\alpha}^{\alpha}\int_0^1 x(r,\theta)\,r\,dr\,d\theta \quad = \quad \int_{-\alpha}^{\alpha}\int_0^1 r^2\,\cos\theta\,dr\,d\theta$$

and since the integrand separates into the product of a function of r and a function of θ (recall the formula that was mentioned in `update002.pdf`), we know that the iterated integral can also be expressed as the following product:

$$\int_{-\alpha}^{\alpha} \cos\theta\,d\theta \;\cdot\; \int_0^1 r^2\,dr\ .$$

Although the problem does not ask for it, the value of the iterated integral is equal to the product of $2\sin\alpha$ (the value of the first term) with $\frac{1}{3}$ (the value of the second term).

See the last page for still further calculations related to this problem.

Further calculations involving problems 2 and 4

The integrals in these problems can be used to find the centers of mass for the respective regions. For problem 2, symmetry considerations show that the center lies on the y-axis, so that $\overline{x} = 0$, while we also know that $\overline{y}$ is the quotient of the computed integral by the area of the region. This area is equal to the integral of $(1-x^4)-(x^2-1) \;=\; 2-x^4-x^2$ from -1 to $+1$, and the latter turns out to be $44/15 \;=\; 132/45$, so that $\overline{y} = 26/132 \;=\; 13/61$.

Similarly, in problem 4 symmetry considerations imply that the center lies on the x-axis, so that $\overline{y} = 0$, while we also know that $\overline{x}$ is the quotient of the computed integral by the area of the region. Since the angle between the two flat sides of the region is 2α, it follows that the area bounded by the region is just α (recall $A = \frac{1}{2}a^2\gamma$, where a is the radius and γ is the angle measurement in radians; in this situation $\gamma = 2\alpha$). Therefore we have that

$$\overline{x} \quad = \quad \frac{2\sin\alpha}{3\alpha}\ .$$