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Mathematics 10B–010, Fall 2009, Examination 2

Answer Key

1. [*25 points*] Using spherical coordinates, set up and evaluate the integral of $f(x, y, z) = 1 - x^2 - y^2 - z^2$ over the disk defined by $x^2 + y^2 + z^2 \leq 1$.

SOLUTION

In spherical coordinates the integrand is $1 - \rho^2$, so using spherical coordinates we may write the integral as

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 (1 - \rho^2)\, \rho^2 \, \sin\phi \, d\rho \, d\phi \, d\theta \ .$$

We now evaluate this from the inside out to see that the threefold iterated integral is equal to

$$\int_0^{2\pi} \int_0^{\pi} \sin\phi \left(\frac{\rho^3}{3} - \frac{\rho^5}{5} \right) \Big|_0^1 \, d\phi \, d\theta \ = \ \int_0^{2\pi} \int_0^{\pi} \sin\phi \cdot \frac{2}{15} \, d\phi \, d\theta \ =$$

$$\frac{2}{15} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} \, d\theta \ = \ \frac{4}{15} \cdot 1 \Big|_{\theta=0}^{\theta=2\pi} \ = \ \frac{8\pi}{15} \ .$$

2. [*20 points*] Let Γ be the piece of the hyperbola $y = 1/x$ starting at $(1,1)$ and ending at $(2, \frac{1}{2})$. Evaluate the line integral

$$\int_\Gamma y\,dx \;-\; x\,dy\ .$$

[*Hint:* Use the parametrization $(t, 1/t)$.]

SOLUTION

If we use the indicated parametrization and substitute into the formula for expressing the line integral as an ordinary integral, we see that the line integral is equal to

$$\int_1^2 \frac{1}{t}\,dt \;-\; t\left(-\frac{dt}{t^2}\right) \;=\; \int_1^2 \frac{dt}{t} \;+\; \frac{dt}{t} \;=\; 2\cdot\log_e t\,\Big|_1^2 \;=$$

$$2\log_e 2 \;=\; \log_e 4\ .$$

3. [*25 points*] Let Γ be the boundary of the square S defined by $0 \leq x, y \leq 1$, and take a counterclockwise parametrization for Γ. Find a continuous function $k(x,y)$ such that the line integral $\int_\Gamma (x^2 - y^2)\,dx \; + \; 2xy\,dy$ is equal to $\int_0^1 \int_0^1 \; k(x,y)\,dy\,dx$, and evaluate this integral.

SOLUTION

By Green's Theorem the line integral is equal to

$$\int_0^1 \int_0^1 \left(\frac{\partial}{\partial x}(2xy) \; - \; \frac{\partial}{\partial y}(x^2+y^2) \right) dy\,dx \quad =$$

$$\int_0^1 \int_0^1 \big(2y \; - \; (-2y) \big)\,dy\,dx \quad = \quad \int_0^1 \int_0^1 \; 4y\,dy\,dx$$

so that $k(x,y) = 4y$. If we perform the iterated integrals from the inside out, we see that the right hand side is equal to

$$\int_0^1 \; 2y^2 \Big|_0^1 \; dx \quad = \quad \int_0^1 \; 2\,dx \quad = \quad 2\;.$$

4. [*30 points*] In each of the following examples, determine whether the vector field $\mathbf{F}$ is the gradient of some function g, and if so find all functions g such that $\nabla g = \mathbf{F}$:

(*a*) $\mathbf{F}(x, y, z) = (-y, x, 3xz^2)$

(*b*) $\mathbf{F}(x, y) = (2xy, x^2 + y^2)$

SOLUTION

To simplify notation, denote the coordinate functions of the vector field as usual by P, Q and (in the 3-dimensional case) R.

For (a), the vector field **is not** a gradient because

$$\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial R}{\partial x} = 3z^2 \neq 0 = \frac{\partial Q}{\partial z}.$$

For (b), the vector field **is** a gradient because

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y}.$$

To find the potential function g, start with

$$g(x, y) = \int 2xy\, dx = x^2 y + h(y)$$

for some function of integration $h(y)$. Since the second partial derivative if g must be $x^2 + y^2$, the same is true if we partially differentiate the right hand side of the displayed equation. But this yields $x^2 + h'(y)$, and since this is $x^2 + y^2$ we conclude that $h'(y) = y^2$, so that $h(y) = \frac{1}{3}y^3 + C$ for some constant of integration C. If we substitute this into the displayed equation, we see that

$$g(x, y) = x^2 y + \frac{y^3}{3} + C$$

where C is a constant of integration.