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Mathematics 10B–010, Fall 2009, Examination 3

Answer Key

1. [*15 points*] Given the surface with parametrization $\mathbf{X}(u, v) = (u + v, u - v, v)$, find a function $f(x, y)$ such that the surface satisfies the equation $z = f(x, y)$. [*Hint:* The surface is a plane.]

SOLUTION

The parametrization yields $x = u + v$, $y = u - v$ and $z = v$. We already have z expressed in terms of u and v, and if we substitute $z = v$ into the second equation we find that $y = u - z$, so that $u = y + z$. We now have u and v in terms of y and z, and if we substitute this into the first equation we find that $x = u + v = y + 2z$, or equivalently $x - y - 2z = 0$.

2. [*15 points*] Set up (but do not evaluate) an iterated integral formula for the area of the surface $z = e^x \sin y$ where $0 \leq x \leq 1$ and $0 \leq y \leq \pi$. The integrand should be expressed as a specific function, and all limits of integration should also be described explicitly.

SOLUTION

Let $f(x,y) = e^x \sin y$. Then the standard normal vector $\mathbf{N}(x,y)$ for the surface $z = e^x \sin y$, which is the graph of f, is given by $(-f_1, -f_2, 1)$, which is equal to

$$(-e^x \sin y, -e^x \cos y, 1)$$

and its length is given by

$$|\mathbf{N}(x,y)| \quad = \quad \sqrt{1 + e^{2x}} \ .$$

The area of the surface is given by the integral of this function over the domain on which the function is defined, and therefore it is equal to

$$\int_0^1 \int_0^\pi \ \sqrt{1+e^{2x}} \ dy\, dx \ .$$

3. [*20 points*] The parametrized curve $\Gamma(t) = (\cos t, \sin t, \cos 2t)$, where $0 \leq t \leq 2\pi$, bounds the portion of the surface $z = x^2 - y^2$ which lies inside the cylinder $x^2 + y^2 \leq 1$. Use Stokes' Theorem to express the line integral

$$\int_\Gamma y^3\,dx \ + \ 4xy^2\,dy \ + \ e^{z^2}\,dz$$

as a double integral over the planar region defined by $x^2 + y^2 \leq 1$. You need not evaluate this integral.

SOLUTION

Let S be the surface which Γ bounds. The integrand in the line integral can be written as $\mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = (y^3,\, 4xy^2,\, e^{z^2})$, and thus by Stokes' Theorem the line integral equals

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{S}\ .$$

If we use the standard 3×3 determinant formula for the curl, we see that

$$\nabla \times \mathbf{F} \ = \ \left(D_y e^{z^2} - D_z 4xy^2,\, D_z y^3 - D_x e^{z^2},\, D_x 4xy^2 - D_y y^3 \right) \ = \ (0, 0, y^2)\ .$$

Also, the standard upward normal is given by $\mathbf{N}(x, y) = (-z_x, -z_y, 1) = (-y, -x, 1)$, so that $(\nabla \times \mathbf{F}) \cdot \mathbf{N} = y^2$ and hence the surface integral is equal to

$$\int\int_D y^2\,dy\,dx$$

where D is the disk over which the parametrization is defined; namely D is the planar region defined by $x^2 + y^2 \leq 1$.

This integral can be evaluated directly, but the problem does not require the evaluation of the integral.

4. [*20 points*] Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x, y, z)$ and S is the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant. Take the outward pointing normal to S.

SOLUTION

We shall use spherical coordinates to parametrize the surface as

$$\mathbf{X}(\theta, \phi) \quad = \quad (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

where $0 \leq \theta, \phi \leq \frac{1}{2}\pi$. It follows immediately that $\mathbf{F}(\theta, \phi) = \mathbf{X}(\theta, \phi)$, and since the surface is the unit sphere, the length of $\mathbf{F}$ is always 1.

The standard normal vector to this parametrization is equal to

$$\mathbf{N}(\theta, \phi) \quad = \quad \sin\phi \cdot (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \quad = \quad \sin\phi \cdot \mathbf{F}$$

and it follows that

$$\mathbf{F} \cdot \mathbf{N} = \sin\phi |\mathbf{F}|^2 = \sin\phi \ .$$

Therefore the surface integral is equal to

$$\int_0^{\pi/2} \int_0^{\pi/2} \sin\phi \, d\theta \, d\phi \quad = \quad \int_0^{\pi/2} \frac{\pi}{2} \sin\phi \, d\theta \, d\phi \quad =$$

$$\frac{\pi}{2} - \cos\phi \Big|_0^{\pi/2} \quad = \quad \frac{\pi}{2} \ .$$

5. [*20 points*] Let S be the surface $z = \exp(1 - x^2 - y^2)$ where $0 \leq x^2 + y^2 \leq 1$, and take the upward pointing normal to S. If $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = (x, y, 3 - 2z)$, explain why $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ is equal to $\int\int_T \mathbf{F} \cdot d\mathbf{T}$, where T is the surface $z = 1$ with the upward pointing normal , and as before $0 \leq x^2 + y^2 \leq 1$. Using this, compute

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \ .$$

Note. If A is a real valued expression, then $\exp(A) = e^A$.

SOLUTION

If D is the region defined by $0 \leq x^2 + y^2 \leq 1$ and $1 \leq z \leq \exp(1 - x^2 - y^2)$, then the boundary of S with the outward normal orientation is given by the union of S and $-T$, where $-T$ is equal to T with the **downard** pointing normal. Therefore we have

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \ - \ \int\int_T \mathbf{F} \cdot d\mathbf{T} \ = \ \int\int\int_D \nabla \cdot \mathbf{F} \, dV \ .$$

Now $\nabla \cdot \mathbf{F} = 1 + 1 - 2 = 0$, and therefore

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \ = \ \int\int_T \mathbf{F} \cdot d\mathbf{T} \ .$$

Thus it is only necessary to compute the right hand side. If we take the standard parametrization $(x, y, 1)$ for T, the upward pointing normal $\mathbf{N}(x, y)$ is given by $(0, 0, 1)$, so that $\mathbf{F} \cdot \mathbf{N} = 3 - 2z$, which is equal to 1 since $z = 1$ on the surface T. Therefore the surface integral over T is given by the ordinary integral

$$\int\int_D 1 \, dA \ = \ \text{Area}(D) \ = \ \pi \ .$$

6. [*10 points*] Suppose that S is a closed piecewise smooth surface bounding a region D in coordinate 3-space, take the outward pointing normal for S, and let $\mathbf{F}$ be the vector field $(2x, 3y, 4z)$. Find a constant k such that

$$\text{Volume}(D) \quad = \quad \frac{1}{k} \int\int_S \mathbf{F} \cdot d\mathbf{S} \ .$$

SOLUTION

The divergence of the vector field is equal to $2 + 3 + 4 = 9$, and therefore by the Divergence Theorem we have

$$\frac{1}{k} \int\int_S \mathbf{F} \cdot d\mathbf{S} \quad = \quad \frac{1}{k} \int\int\int_D \nabla \cdot 9 \, dV \quad =$$

$$\frac{9 \cdot \text{Volume}(D)}{k} \quad = \quad \frac{9}{k} \cdot \text{volume}(D) \ .$$

We need the value of k such that $1 = 9/k$, and this means that $k = 9$.