Flux integrals in the plane and Green's Theorem

As a transition from Section 7.2 to Section 7.3 of the course text, it is worthwhile to go back and discuss something from Section 6.2 which is related to our concept of flux integrals in the plane.

Given a vector field $\mathbf{F} = (P, Q)$ defined in an open region U in the plane and a curve Γ in that region with standard preferred normal direction

$$\mathbf{N}(t) \quad = \quad \big(-y'(t),\, x'(t)\big)$$

we defined the **flux** of the vector field to be the line integral

$$\int_\Gamma (\mathbf{F} \cdot \mathbf{N})\, ds \quad = \quad \int_\Gamma \mathbf{F} \cdot d\mathbf{N} \quad = \quad \int_\Gamma -Q\, dx \; + \; P\, dy \; .$$

Suppose now that we have a simple closed curve Γ in the region such that Γ bounds a subregion D inside U. Then, as noted in Section 6.2 of the course text, Green's Theorem yields the following identity involving the flux integral over Γ and a double integral over the region D:

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{N} \quad = \quad \int\!\!\int_D (\nabla \cdot \mathbf{F})\, dA \quad = \quad \int\!\!\int_D \text{divergence}(\mathbf{F})\, dA$$

One objective of Section 7.3 is to formulate an analog of this identity for flux integrals of vector fields over surfaces in 3-dimensional space.