An example involving the Mean Value Theorem for Integrals

PROBLEM. Find the average value of the function xyz over the cube $0 \leq x, y, z \leq 3$, and find a point (x_0, y_0, z_0) where this average value is realized.

Solution. The volume of the cube is $3^3 = 27$, so the average value is equal to

$$\frac{1}{27} \cdot \int_0^3 \int_0^3 \int_0^3 xyz\,dz\,dy\,dx \ .$$

Since the integrand factors into a product of a function of x, a function of y, and a function of z, it follows that the integral is equal to

$$\int_0^3 x\,dx \cdot \int_0^3 y\,dy \cdot \int_0^3 z\,dz \quad = \quad \left(\int_0^3 x\,dx \right)^3 \quad =$$

$$\left(\frac{x^2}{2} \Big|_0^3 \right)^3 \quad = \quad \left(\frac{9}{2} \right)^3 \ .$$

It follows that the average value of xyz is equal to

$$\frac{1}{27} \cdot \left(\frac{9}{2} \right)^3 \quad = \quad \frac{27}{8} \ .$$

Any choice of (x_0, y_0, z_0) for which $x_0 y_0 z_0 = 27/8$ will realize the average value. The simplest choice is to take all of them equal to $3/2$.