ADDITIONAL EXAMPLES FOR CHANGE OF VARIABLES

The following sort of problem appears in many multivariable calculus texts, including the one for this course.

Find the double integral

$$\int \int_R (x+y)^{1/2}\,(x-y)^{1/3}\,dA$$

where R is the region bounded by the lines $y = x$, $y = -x$, $y = x - 1$, $y = 2 - x$.

One way to start analyzing this problem is to sketch the region R. It turns out that R is the solid rectangular region for the rectangle with vertices $(0,0)$, $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$, and $(1,1)$. However, the real point of such an exercise is that one wants to make a linear change of variables to simplify the integrand. The natural choice is $u = x + y$, $v = x - y$.

Once we do this, we have to determine the region S in the $uv-$plane which maps to R in the $xy-$plane. The first step in this process is to obtain the formulas for x and y in terms of u and v. Here are the results:

$$x \quad = \quad \tfrac{1}{2}(u+v)\ , \quad y \quad = \quad \tfrac{1}{2}(u-v)$$

We can find the boundary curves for the corresponding region S in the $uv-$plane by substituting these formulas for x and y in terms of u and v. If we carry out these substitutions we see that the line $y = x$ comes from the line $v = 0$, the line $y = -x$ comes from the line $u = 0$, the line $y = x - 1$ comes from the line $v - 1$, and the line $y = 2 - x$ comes from the line $u = 2$. The region in the $uv-$plane bounded by these lines is the rectangular region defined by $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

So now we know the set over which the $du\,dv$ integral is to be evaluated. In order to complete the Change of Variables formula in this case we need to compute the Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and we can do this fairly easily using the formulas for x and y in terms of u and v. The Jacobian turns out to be the following constant:

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \quad = \quad -\frac{1}{4} - \frac{1}{4} \quad = \quad -\frac{1}{2}$$

It follows that the absolute value of this Jacobian is $\frac{1}{2}$.

Putting everything together, we see that the original double integral is equal to the double integral

$$\int\int_S u^{1/2}\, v^{1/3}\, \tfrac{1}{2}\, dA$$

and the latter in turn is equal to the following iterated integral:

$$\int_0^1 \int_0^2 u^{1/2}\, v^{1/3}\, du\, dv$$

Direct computation shows that this integral is equal to

$$\int_0^2 u^{1/2}\, du \;\cdot\; \int_0^1 v^{1/3}\, dv \;=\; \frac{2}{3} \cdot 2^{3/2} \cdot \frac{3}{4} \;=\; 2^{1/2}\ .$$

In principle one could work this problem without using change of variables, but the integrands would be much messier to manipulate. However, one does not quite get something for nothing because there is clearly a significant amount of work needed to set up the change of variables properly.■

Here is another example using polar coordinates:

Find the integral of the function $f(x, y) = y$ over the solid semicircular region R in the upper half plane ($y \geq 0$) bounded by the circle $x^2 + y^2 = 1$ and $y = 0$.

The first thing is to determine the corresponding region S in polar coordinates. This is given by the conditions $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$.

Next, we have to express y as a function of r and θ, which is easy because we know that $y = r \sin\theta$.

We now have to compute the Jacobian of (x, y) with respect to (r, θ). This derivation appears at the bottom of page 331 in the text, and the Jacobian turns out to be r.

Combining these with the Change of Variables formula, we see that the original integral is equal to

$$\int\int_S (r \sin\theta) \cdot r\, dr\, d\theta \;=\; \int_0^\pi \int_0^1 r^2 \sin\theta\, dr\, d\theta \;=$$

$$\int_0^\pi \sin\theta\, d\theta \;\cdot\; \int_0^1 r^2\, dr \;=\; 2 \;\cdot\; \frac{1}{3} \;=\; \frac{2}{3}\ .$$

As indicated in Section 5.6, the integral of the function y over the given region is one term in the formula for the solid semicircle's center of mass.■

Finally, we shall give an example involving spherical coordinates. As noted on page 339 of the text, the absolute value of the Jacobian is given by

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| \quad = \quad \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| \quad = \quad \rho^2 \sin\phi \ .$$

The equality of the first two terms follows because the Jacobian changes signs if one interchanges a pair of variables in either the numerator or denominator; this in turn comes from the fact that a determinant changes sign if one switches two rows or two columns.

Here is the statement of the problem:

Set up the integrals of the functions $f(x, y, z) = y$ and $f(x, y, z) = z$ over the solid region R in the first octant between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. It is not necessary to evaluate the integrals.

By definition, the first octant is the set of all points (x, y, z) such that all three coordinates are nonnegative.

We first need to describe the given region in terms of ρ, θ and ϕ. The inequalities $1 \leq \rho \leq 2$ are given, and for points in the first octant the longitude θ ranges between 0 and $\frac{1}{2}\pi$. Also, in the first octant the modified latitude ϕ also lies between these two limits.

The standard formulas for spherical coordinates imply that $z = \rho\cos\phi$ and $y = \rho\sin\theta\sin\phi$. Also, we already noted that the absolute value for the Jacobian of (x, y, z) with respect to (ρ, θ, ϕ) is equal to $\rho^2 \sin\phi$.

If we put all this together with the change of variables formula, we obtain the following iterated integral formulas:

$$\int\int\int_R y\, dV \quad = \quad \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^3 \sin^2\phi \sin\theta \, d\rho\, d\theta\, d\phi$$

$$\int\int\int_R z\, dV \quad = \quad \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^3 \sin\phi \cos\phi \, d\rho\, d\theta\, d\phi$$

Note that each of these iterated integrals can be rewritten as a product of three ordinary integrals with respect to the variables ρ, θ and ϕ.

We shall not give the details of the computations here. However, one can use the material on centers of mass in Section 5.6 to conclude that the values of these two somewhat different looking integrals must be equal. By the formulas for center of mass $(\overline{x}, \overline{y}, \overline{z})$ of the region described in the problem, the integrals in the problem are equal to $\overline{y} \cdot W$ and $\overline{z} \cdot W$, where W is the volume of the region. However, since the defining equations for the region are symmetric in x, y and z, it follows that the coordinates of the center of mass must all be equal! Therefore the two integrals in the problem must be equal. This fact can be useful in checking the correctness of answers, for if one obtains different values for the two integrals, then there must be a computational mistake somewhere.■