A path dependent line integral

In the lectures we noted that a vector field $\mathbf{F}(x,y) = \big(P(x,y),\, Q(x,y)\big)$ defined on a suitably nice (more precisely, hole-free or simply connected) region of the plane was a gradient if and only if its coordinates satisfy the compatibility condition

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x} \; .$$

This condition arises from the equality of mixed second partial derivatives

$$\frac{\partial^2 g}{\partial x \partial y} \quad = \quad \frac{\partial^2 g}{\partial y \partial x}$$

and because of this we know that the compatibility condition holds if $\mathbf{F} = \nabla g$, for some g, regardless of whether or not the region is simply connected.

On the other hand, we also noted that the compatibility condition is not enough to guarantee that $\mathbf{F}$ is a gradient if the region is not simply connected, and we gave the following example, which is defined on the region of all points in the plane except the origin:

$$\mathbf{F}(x,y) \quad = \quad \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

The standard rules for partial differentiation yield the compatibility conditions for P and Q, and in fact we have

$$\frac{\partial P}{\partial y} \quad = \quad \frac{y^2-x^2}{(x^2+y^2)^2} \quad = \quad \frac{\partial Q}{\partial x} \; .$$

However, it turns out that $\mathbf{F}$ is NOT a gradient, and it is worthwhile to look at this more closely (many books covering multivariable calculus explain this, but the course text does not).

Consider the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{s}$ where Γ is the counterclockwise circle with parametric equations $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then the line integral over the closed path Γ is equal to

$$\int_\Gamma \frac{-y}{x^2+y^2}\, dx \; + \; \frac{x}{x^2+y^2}\, dy \quad = \quad \int_0^{2\pi} \frac{(-\sin\theta)(-\sin\theta\, d\theta)}{\sin^2\theta + \cos^2\theta} \; + \; \frac{(\cos\theta)(\cos\theta\, d\theta)}{\sin^2\theta + \cos^2\theta} \quad =$$

$$\int_0^{2\pi} 1 \cdot d\theta \quad = \quad 2\pi \; .$$

Now if $\mathbf{F}$ were a gradient, then the line integral would be the same for all closed paths, and in particular this means it would have to be equal to zero. Since the given line integral is nonzero, we can use *reductio ad absurdum* to conclude that there is no function g on the plane with one point removed for which $\nabla g = \mathbf{F}$.