Computing flux integrals in the plane

PROBLEM. Find the flux of the vector field $\mathbf{F}(x, y) = (x^2, y^2)$ across the line segment joining $(1, 0)$ to $(0, 1)$ where the preferred normal direction points away from the origin.

Solution. First take a standard parametrization of the curve described in the problem: $\gamma(t) = (1-t, t)$. It follows that the tangent vector to this curve is given by $\gamma'(t) = (-1, 1)$. The preferred normal direction associated to this parametrization is given by the **Right Hand Rule**: *If you are facing in the tangent direction, then the preferred normal direction is on your right.*

In other words, if (x', y') is the tangent direction, then $(y', -x')$ is the preferred normal direction.

Specializing to this problem, we see that the preferred normal direction for the given parametrization is $\mathbf{N}(t) = (1, 1)$. This vector indeed points outward from the origin, so out parametrization is OK. If by some chance we ended up with a preferred normal pointing in the wrong direction, one easy way of fixing things would be to take the reverse parametrization of the curve; specifically, if γ is defined on the interval $[a, b]$, then the reverse parametrization is given by $\gamma^*(t) = \gamma(b-t+a)$, and its preferred normal direction will then point in the correct direction.

Solving the given problem is now easy, for we have

$$\mathbf{F} \cdot \mathbf{N} = x^2 + y^2$$

and hence the value of this expression at $\gamma(t)$ is equal to $(1-t)^2 + t^2 = 2t^2 - 2t + 1$. Therefore the flux integral is equal to the ordinary integral

$$\int_0^1 (2t^2 - 2t + 1)\, dt = \frac{2t^3}{3} - t^2 + t \Big|_0^1 = \frac{2}{3}\,.$$

We could also work this problem using the formula $\int_\Gamma -Q\, dx + P\, dy$, where $\mathbf{F} = (P, Q)$. Substitution into this formula yields

$$\int_0^1 (-t^2)\,(-dt) + (1-t)^2\, dt$$

which is exactly the same integral we obtained above.