An example involving Stokes' Theorem

PROBLEM. Find the line integral

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = (x + 2y + 4z, x^2 + y^2 + z^2, x + y + z)$ and Γ is the boundary of the solid triangular region S consisting of all points in the first octant which are on the plane with equation $x + y + z = 1$, taken in the counterclockwise sense.

The advantage of using Stokes' Theorem for this problem is that the line integral has three smooth pieces

(a) the segment joining $(1, 0, 0)$ to $(0, 1, 0)$,
(b) the segment joining $(0, 1, 0)$ to $(0, 0, 1)$,
(c) the segment joining $(0, 0, 1)$ to $(1, 0, 0)$,

and accordingly the evaluation of the line integral involves three separate integral calculations. Stokes' Theorem says the line integral equals a surface integral which can be computed without breaking things up into smaller pieces.

Solution. The first thing we should do is find the parametrization for the solid triangle S; we can do this by using the equation $z = 1 - x - y$ to write $\mathbf{X}(x, y) = (x, y, 1 - x - y)$. It follows immediately that the normal vector is given by $\mathbf{N}(x, y) = (-z_x, -z_y, 1) = (1, 1, 1)$. Note that the parameter values run over all x and y in the first quadrant such that $0 \leq x + y \leq 1$, so that $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$.

Next, we need to compute the curl of $\mathbf{F}$ using the standard determinant formula:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ x + 2y + 4z & x^2 + y^2 + z^2 & x + y + z \end{vmatrix} = (1 - 2z, 3, 2x - 2)$$

Therefore $(\nabla \times \mathbf{F}) \cdot \mathbf{N} = 2x - 2z + 2 = 4x + 2y$. This means that the line integral and equivalent surface integral are given by

$$\int_0^1 \int_0^{1-x} 2y + 4x \; dy \, dx = \int_0^1 (1 - x)^2 + 4x(1 - x) \, dx =$$

$$\int_0^1 2x - 3x^2 + 1 \, dx = x^2 - x^3 + x \Big|_0^1 = 1 .$$