Solution to supplementary exercise 7.3.S1

The curve Γ is the triangle with vertices $(0,0,0)$, $(0,2,0)$ and $(1,1,1)$. In order to apply Stokes' Theorem, we need to find a surface S which it bounds. Since the problem did not say anything about the sense of the curve, there are two possible answers depending upon the choice of sense, and each is the negative of the other.

Once we find the surface S and its normal $\mathbf{N}$, then by Stokes' Theorem we know that the line integral is equal to $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ or equivalently to $\int\int_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dA$, where D is the domain of the parametrization for S.

The region S bounded by Γ is the solid triangular region definable parametrically by

$$\mathbf{X}(u,v) \quad = \quad v(0,2,0) \ + \ u(1,1,1)$$

where $0 \le u, v \le 1$ and $u+v \le 1$. It follows that the standard normal vector $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2$ for this parametrization is equal to

$$(1,1,1) \ \times \ (0,2,0) \quad = \quad (-2,0,2)$$

(compute this for yourself!). Furthermore, since the vector field $\mathbf{F}$ is given by $(2y, 3z, x)$, its curl is equal to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_x & D_y & D_z \\ 2y & 3z & x \end{vmatrix} \quad = \quad (-3,-1,-2)$$

so that $(\nabla \times \mathbf{F}) \cdot \mathbf{N} \ = \ (-2,0,2) \cdot (-3,-1,-2) \ = \ 6 - 4 = 2$. This means that the line integral is equal to $\int\int_D 2\, dA$, where D is the set of all (u,v) in the coordinate plane such that $u, v \ge 0$ and $u + v \le 1$; in other words, D is the solid isosceles right triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The area of this triangle is $\frac{1}{2}$ and therefore the value of the integral is equal to 1.