Answers to additional exercises for Colley, Chapter 5

Section 5.1

S1. 3

S2. 8

Section 5.2

S1. 32

S2. **TRUE.** Everything about the integral on the left hand side is symmetric with respect to both the $x-$axis and the $y-$axis. The first of these implies that the integral over the square defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ is twice the integral over the rectangle define by $0 \leq x \leq 1$ and $-1 \leq y \leq 1$, and the second implies that the integral over the latter rectangle is equal to twice the integral over the square defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Combining these, we conclude that the integral over the larger square is four times the integral over the smaller one.

S3. $\log_e 3 - \frac{3}{2} \log_e 2$

S4. $\dfrac{1}{3}$

S5. $\dfrac{2}{3}$

S6. The region A is given by the inequalities $0 \leq y \leq \sqrt{3}$ and $-y \leq x \leq 2y - y^2$. In order to interchange the order of integration, it is necesary to split this up into two regions, one of which is defined by $0 \leq x \leq 1$ and $1 - \sqrt{1-x} \leq y \leq 1 + \sqrt{1-x}$, and the other of which is defined by $-\sqrt{(3)} \leq x \leq 0$ and $-x \leq y \leq 1 + \sqrt{1-x}$. The area is equal to

$$\frac{9}{2} \; - \; \sqrt{3}\ .$$

S7. 1

Section 5.3

S1. $\displaystyle\int_0^1 \int_y^1 \sqrt{1-x^2}\, dx\, dy$

S2.

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \log_e(1 + x^2y^2)\, dy\, dx$$

S3.

$$\int_0^1 \int_{x^2}^{\sqrt[3]{x}} \tan(xy)\, dy\, dx$$

S4. $\frac{1}{2}(\cos 1 - 1)$, where the θ in $\cos\theta$ is given by radian measure.

Section 5.4

S1. 12

S2. $\dfrac{3}{8}$

S3. $\dfrac{(e-1)^2}{2}$

S4. The region is defined by the inequalities $-1 \le z \le 1$, $-\sqrt{(1-z^2)/2} \le x \le \sqrt{(1-z^2)/2}$, and $-\sqrt{1-x^2-z^2} \le y \le \sqrt{1-x^2-z^2}$.

S5. The region is defined by the inequalities $0 \le x \le 4$, $0 \le y \le 4-x$, and $0 \le z \le x+y+1$.

S6. $\dfrac{7}{60}$

S7. $\displaystyle\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

Section 5.5

S1. The corresponding region in the $uv-$plane is the triangle whose vertices are $(0,0)$, $(0,1)$ and $(1,0)$.

S2. The corresponding region in the $uv-$plane is the square whose vertices are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

S3. $\dfrac{a^3}{3}$

S4. $\dfrac{2}{3}$

S5. 0

S6. $\dfrac{\pi}{8}$

S7. $\dfrac{8}{3}$

S8. 0

S9. $uv(u+v-uv)$

S10. 17

S11. $\dfrac{\pi}{2}$ is the value of the original integral. The Jacobian of (x,y) with respect to (u,v) turns out to be $4(u^2+v^2)$, so the correspondint uv-integrals will be given as follows:

$$\int\int_R x\,dx\,dy \quad = \quad \int\int_S (u^2-v^2)\cdot 4(u^2+v^2)\,du\,dv$$

$$\int\int_R y\,dx\,dy \quad = \quad \int\int_S (2uv)\cdot 4(u^2+v^2)\,du\,dv$$

For computational purposes it is convenient to rewrite the right hand sides in polar coordinates. If make such a change of variables, these are the iterated integrals we obtain (in order):

$$\int_0^{\pi/2}\int_0^1 4r^5(\cos^2\theta - \sin^2\theta)\,dr\,d\theta\ , \qquad \int_0^{\pi/2}\int_0^1 8r^5(\cos\theta\sin\theta)\,dr\,d\theta$$

Section 5.6

S1. $\overline{x} = \overline{y} = 0$, $\overline{z} = h/4$

S2. $\overline{x} = \overline{y} = 0$, $\overline{z} = 3r/8$

S3. $\overline{x} = \overline{y} = 0$,

$$\overline{z} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

S4. $\frac{1}{2}(1 + e + e^{-1})$

S5. $\dfrac{3}{\pi}$