Answers to additional exercises for Colley, Chapter 6

Section 6.1

S1. $\dfrac{12\pi^2\sqrt{\pi}}{2}$

S2. 10

S3. 49,920

S4. $\dfrac{35}{6}$

S5. 2

S6. $-10\pi^2$

S7. The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same as the ordinary integral of $\mathbf{F} \cdot \mathbf{r}'$ over the interval on which the parametrization is defined. Now if $\mathbf{F}$ is perpendicular to the curve then this dot product is zero, so the integral must be zero. On the other hand, if $\mathbf{F}$ is parallel to the curve so that $\mathbf{F} = \lambda\mathbf{r}'$, then the integral reduces to the ordinary integral of $\lambda\,|\mathbf{r}'|^2$. But $\lambda\,|\mathbf{r}'|$ is just the length of $\mathbf{F}$, and $|\mathbf{r}'(t)|\,dt = ds$, so if we substitute these into the integral expression we conclude that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same as $\int_C |\mathbf{F}| \cdot ds$.

Section 6.2

S1. $\dfrac{4}{3}$

S2. $\dfrac{32}{3}$

S3. It suffices to verify that the line integrals in the equations are equal to $2\int\int_A x\,dA$ and $2\int\int_A y\,dA$ respectively. These follow by direct substitution into the formula

$$\int_C M\,dx \;+\; N\,dy \quad = \quad \int\int_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\;.$$

S4. By Green's Theorem we have

$$\int_{\partial D} \left(\frac{\partial f}{\partial y}\right) dx \;-\; \left(\frac{\partial f}{\partial x}\right) dy \quad = \quad \int\int_D \left[-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \;-\; -\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] dx\,dy \quad =$$

$$-\int\int_D \left(\nabla^2 f\right)\,dx\,dy$$

and since we are assuming $\nabla^2 f = 0$ it follows that the integrand, and hence also the integral, must be zero.

Section 6.3

S1. $f(x,y) = x^2y + \frac{1}{3}y^3$

S2. No such vector field exists because the partial derivative of the first coordinate with respect to y is unequal to the partial derivative of the second coordinate with respect to x (in fact, a similar situation holds if we take an arbitrary pair of coordinates for the vector field on the right hand side of the equation!).

S3. $f(x,y) = xz + 2xy - yz$

S4. 0. One can use the fact that the associated vector field $\mathbf{F}$ is the gradient of xyz, or equivalently one can simply compute that the integrand is zero over each smooth piece of the broken line curve.

S5. In this case the vector field corresponding to the integrand is equal to the gradient of $z^3x + x^2y$.

S6. First of all, the function f is equal to $yz\,e^{x^2} + K$ for some constant K. Since $f(0,0,0) = 5$, it follows that $K = 5$, so that $f(1,1,2) = 2e + K = 5 + 2e$.