Answers to additional exercises for Colley, Chapter 7

Section 7.1

S1. $x - y - 2z = 0$

S2. $\mathbf{x}(u,v) = (\sin u \cos v, \sin u \sin v, u), (0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi)$

S3. $(2 - \sqrt{2})\pi$

S4. The tangent plane is $z = x + y$ (which is also an equation satsified by the surface, although the latter only consists of points whose coordinates are nonnegative).

S5. The parametrization is regular for all (u, v).

S6. The parametrization is regular for all $(u, v) \neq (1, -1)$.

Section 7.2

S1. $\sqrt{14}$

S2. $\dfrac{(5\sqrt{5} - 1)\pi}{6}$

S3. $\dfrac{2\pi}{3} \cdot \left(17\sqrt{17} - 1\right)$

S4. $\dfrac{16k\pi\sqrt{5}}{3}$

S5. $-\dfrac{4}{3}$

S6. $\dfrac{243\pi}{2}$

S7. 32π

Section 7.3

S1. 1

S2. -9

S3. $\dfrac{a^2}{4} \cdot (2a^2 + a - 4)$

S4. 0

S5. -16π

S6. We shall use Exercise 24 on page 222 of the text (formula for the curl of a cross product). This identity yields

$$\nabla \times (f\,\nabla g) \quad = \quad f \cdot \nabla \times (\nabla g) \; + \; (\nabla f) \times (\nabla g)$$

and since $\nabla \times (\nabla g) = 0$ by Theorem 3.4.3 on page 218, it follows that the first term on the right hand side drops out. This leaves us with just the second term on the right hand side, and hence we have the identity we wanted to verify.

S7. π

S8. $\dfrac{2\pi}{5}$

S9. For the region under consideration, $\nabla \times \mathbf{F} = 0$ implies $\mathbf{F} = \nabla f$ for some f. But then

$$\nabla^2 f \quad = \quad \nabla \cdot (\nabla f) \quad = \quad \nabla \cdot \mathbf{F}$$

which is assumed to be zero. Therefore $\nabla^2 f = 0$ as required.

S10. The flux integral for $\mathbf{F}$ is equal to the scalar surface integral of the dot product $\mathbf{F} \cdot \mathbf{N}$, where $\mathbf{N}$ is the normal to the surface. If $\mathbf{F}$ is always tangent to the surface, then it is always perpendicular to $\mathbf{N}$ and hence $\mathbf{F} \cdot \mathbf{N} \ = \ 0$, which means that the surface integral must also be equal to zero.

Section 7.4

S1. Since λ is an integrating factor for $\mathbf{F}$, we have $\lambda\mathbf{F} = \nabla g$ for some function g. Since the curl of a gradient is zero, this means that

$$\mathbf{0} \quad = \quad \nabla \times (\nabla g) \quad = \quad \nabla \times (\lambda \mathbf{F}) \quad = \quad (\nabla \lambda) \times \mathbf{F} \ + \ \lambda \, (\nabla \times \mathbf{F}) \ .$$

Since λ is never zero we may rewrite this as

$$\nabla \times \mathbf{F} \quad = \quad \frac{-1}{\lambda} \, (\nabla \lambda) \times \mathbf{F} \ .$$

The right hand side of this equation is orthogonal to $\mathbf{F}$, so the left hand side is also orthogonal to $\mathbf{F}$.

S2. Direct calculation implies that $\nabla \times \mathbf{F} = (-1, -1, -1)$, and therefore $(\nabla \times \mathbf{F}) \cdot \mathbf{F} = -y - z - x$. If all three of x, y, z are positive, then this dot product is negative. By the preceding exercise, it follows that one cannot find an integrating factor for $\mathbf{F}$ on the specified region. In fact, one can show that there is no integrating factor for $\mathbf{F}$ on the **any** nonempty region of 3-dimensional coordinate space.

S3. Here is the formula from Exercise 15(a):

$$\nabla \times (\nabla \times \mathbf{F}) \quad = \quad \nabla(\nabla \cdot \mathbf{F}) \ - \ \nabla^2 \mathbf{F}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$ and $\nabla \cdot \mathbf{F} = 0$, then the displayed formula reduces to

$$\mathbf{0} \quad = \quad \nabla \times \mathbf{0} \quad = \quad \nabla 0 \ - \ \nabla^2 \mathbf{F} \quad = \quad \mathbf{0} \ - \ \nabla^2 \mathbf{F}$$

which in turn means that $\nabla^2 \mathbf{F} = \mathbf{0}$.