ADDITIONAL EXERCISES FOR MATHEMATICS 10B

FALL 2009

Chapter 6

The exercises are organized by sections of the course text (Colley, *Vector Calculus*, Third Edition).

Exercises for Colley, Section 6.1

S1. Evaluate the given line integral over the indicated curve:

$$\int_C \cos z \, ds \qquad C: \ \mathbf{r}(t) \ = \ (\sin t, \cos t, t), \qquad 0 \leq t \leq 2\pi$$

S2. Evaluate the given line integral over the indicated curve:

$$\int_C (x - y) \, ds \qquad C: \ \mathbf{r}(t) \ = \ (4t, \, 3t), \qquad 0 \leq t \leq 2$$

S3. Evaluate the given line integral over the indicated curve:

$$\int_C xyz \, ds \qquad C: \ \mathbf{r}(t) \ = \ (3, \, 12t, \, 5t), \qquad 0 \leq t \leq 2$$

In each of exercises S4 and S5 below, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}(t)$ is the chosen parametrization for the curve C:

S4. $\mathbf{F}(x, y) = (xy, \, y), \qquad \mathbf{r}(t) = (4t, \, t), \quad 0 \leq t \leq 1.$

S5. $\mathbf{F}(x, y) = (3x, \, 4y), \qquad \mathbf{r}(t) = (2\cos t, \, 2\sin t), \quad 0 \leq t \leq \frac{\pi}{2}.$

S6. Evaluate the line integral

$$\int_C x \, dx \ + \ y \, dy \ + \ 5z \, dz$$

where C is parametrized by $\mathbf{r}(t) = (2\cos t, \, 2\sin t, \, t)$, where $0 \leq t \leq 2\pi$.

S7. Suppose we are given a vector field $\mathbf{F}$ and a curve C in the domain over which $\mathbf{F}$ is defined. Explain why $\int_C \mathbf{F} \cdot d\mathbf{s}$ is zero if the tangent vectors to C are perpendicular to $\mathbf{F}$ everywhere and $\int_C \mathbf{F} \cdot d\mathbf{s} \ = \ \int_C |\mathbf{F}| \, ds$ if the tangent vector to C are parallel to $\mathbf{F}$ everywhere.

Exercises for Colley, Section 6.2

S1. Using Green's Theorem, evaluate the line integral

$$\int_C (y-x)\,dx \;+\; (2x-y)\,dy$$

if C is the boundary curve for the (bounded) region lying between the graphs of $y = x$ and $y = x^2 - x$.

S2. Using Green's Theorem, find the area of the region bounded by the graphs of $y = 2x+1$ and $y = 4 - x^2$.

S3. Using Green's Theorem, show that the centroid of a region with area $|A|$ and boundary curve C has the following coordinates:

$$\overline{x} \;=\; \frac{1}{2|A|}\int_C x^2\,dy \qquad\qquad \overline{y} \;=\; \frac{1}{2|A|}\int_C -y^2\,dx$$

S4. Suppose that the function f (with continuous second partial derivatives) is **harmonic** on the region D; in other words

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial x^2} \;+\; \frac{\partial^2 f}{\partial y^2} \;=\; 0\ .$$

Suppose that the boundary of D is a simple closed curve, and call this curve ∂D. Prove that

$$\int_{\partial D} \left(\frac{\partial f}{\partial y}\right) dx \;-\; \left(\frac{\partial f}{\partial x}\right) dy \;=\; 0\ .$$

Exercises for Colley, Section 6.3

S1. Find all functions f such that $\nabla f(x,y) = (2xy,\, x^2+y^2)$. [*Note.* The phrase "all functions" includes the possibility that there are no such functions at all.]

S2. Find all functions f such that $\nabla f(x,y,z) = (-y,\, x,\, 3xz^2)$. [*Note.* The phrase "all functions" includes the possibility that there are no such functions at all.]

S3. Find all functions f such that $\nabla f(x,y,z) = (z+2y,\, 2x-z,\, x-y)$. [*Note.* The phrase "all functions" includes the possibility that there are no such functions at all.]

S4. Evaluate the line integral

$$\int_C yz\,dx \;+\; xz\,dy \;+\; xy\,dz$$

where C is the broken line curve which goes first from $(1,0,0)$ to $(0,1,0)$ along a straight line segment, and then from $(0,1,0)$ to $(0,0,1)$ along a straight line segment.

S5. Let C be the boundary curve for the square $-1 \le x, y \le 1$ parametrized in the usual counterclockwise sense. Prove that

$$\int_C (z^3+2xy)\,dx \;+\; x^2\,dy \;+\; 3xz^2\,dz \;=\; 0\ .$$

S6. Suppose that $f(x,y,z)$ is a function such that $\nabla f = (2xyz\,e^{x^2},\, z\,e^{x^2},\, y\,e^{x^2})$ and $f(0,0,0) = 5$. Find $f(1,1,2)$.