ADDITIONAL EXERCISES FOR MATHEMATICS 10B

FALL 2009

Chapter 7

The exercises are organized by sections of the course text (Colley, *Vector Calculus*, Third Edition).

Exercises for Colley, Section 7.1

S1. Find an equation defining the tangent plane to the parametrized surface $\mathbf{x}(u, v) = (u+v, u-v, v)$ at the point $\mathbf{p} = (1, 1, 0)$. [*Hint:* The first step is to find (u, v) such that $\mathbf{x}(u, v) = \mathbf{p}$).

S2. Give a set of parametric equations for the surface of revolution obtained by revolving the graph of the function $x = \sin z$ $(0 \leq z \leq \pi)$ about the z-axis.

S3. Find the surface area for the piece of the unit sphere $x^2 + y^2 + z^2 = 1$ cut out by the conical region $z \geq \sqrt{x^2 + y^2}$.

Exercises for Colley, Section 7.2

S1. Find the surface area for the portion of the graph of $z = 10 + 2x - 3y$ over the square with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$.

S2. Find the surface area for the portion of the graph of $z = 4 + x^2 - y^2$ over the disk defined by $x^2 + y^2 \leq 1$.

S3. Find the surface area for the portion of the graph of $z = xy$ over the disk defined by $x^2 + y^2 \leq 16$.

S4. A thin conical shell is given in coordinates by $z = 4 - 2\sqrt{x^2 + y^2}$, where $0 \leq z \leq 4$, and the density at each point is proportional to the distance between the point and the $z-$axis (hence is $k\sqrt{x^2 + y^2}$ for some constant k. Find the total mass of this shell.

In each of exercises S5–S7 below, evaluate the surface integral $\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S}$, for the given choices of the vector field (= vector valued function) $\mathbf{F}$ and the oriented surface S. In each case take the upward pointing normal orientation for S.

S5. $\mathbf{F}(x, y, z) = (3z, 4, y)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

S6. $\mathbf{F}(x, y, z) = (x, y, z)$ and S is the portion of the paraboloid $z = 9 - x^2 - y^2$ lying in the upper half-space defined by $z \geq 0$.

S7. $\mathbf{F}(x, y, z) = (x, y, z)$ and S is the portion of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Exercises for Colley, Section 7.3

S1. Using Stokes' Theorem, calculate the line integral

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{x}$$

where $\mathbf{F}(x, y, z) = (2y, 3z, x)$ and Γ bounds the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$, $(1, 1, 1)$.

S2. Suppose that D is the solid triangle consisting of all points which are on the plane with equation $2x+2y+z = 6$ and in the first octant, and let Γ be the boundary of D parametrized in the counterclockwise sense. Evaluate the line integral

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{x}$$

where $\mathbf{F}(x, y, z) = (-y^2, z, x)$.

S3. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x^2 z, -y, xyz)$ and S is the boundary of the cube defined by the inequalities $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$; as usual, take the normal to the surface to point outwards.

S4. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (z + 2x, x^3 y, -2z)$ and S is the boundary of the hemisphere defined by the inequalities $x^2 + y^2 + z^2 \leq 1$ and $0 \leq z \leq 1$; as usual, take the normal to the surface to point outwards.

S5. Evaluate the surface integral $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 16$, the vector field $\mathbf{F}(x, y, z) = (x^2 + y - 4, 3xy, 2xz + z^2)$, and the normal orientation of S is upward.

S6. Suppose that f and g are functions with continuous partial derivatives (defined on a given region). Prove that $\nabla \times (f \nabla g) \ = \ (\nabla f) \times (\nabla g)$.

S7. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where S is the unit sphere with the outward normal and $\mathbf{F}(x, y, z) = (x^3, y^3, z^3)$.

S8. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, wheere S bounds the cylinder defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$, and $\mathbf{F}(x, y, z) = (1, 1, z(x^2 + y^2)^2)$.

S9. Let $\mathbf{F}$ be a vector field defined on all of 3-space such that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$. Prove that $\mathbf{F} = \nabla g$, where g satisfies $\nabla^2 g = 0$ (in other words, g is harmonic).

S10. Suppose that the vector field $\mathbf{F}$ is defined and has continuous partial derivatives on a region containing the closed surface S and the region W bounded by S, and assume further that the restriction of $\mathbf{F}$ to S is tangent to S at all points of the latter. Prove that $\int\int\int_W (\nabla \cdot \mathbf{F}) \, dV \ = \ 0$.

Exercises for Colley, Section 7.4

S1. Let $\mathbf{F}$ be a vector field defined in a region of coordinate 3-space. An *integrating factor* for $\mathbf{F}$ is a positive valued function λ defined on the region such that λ has continuous partial derivatives and $\lambda\mathbf{F} = \nabla g$ for some function g. Prove that if $\mathbf{F}$ has an integrating factor, then $\mathbf{F}$ and $\nabla \times \mathbf{F}$ are perpendicular to each other. [*Hint:* What can we say about the curl of $\lambda\mathbf{F}$? Why is the curl of $\mathbf{F}$ equal to $\nabla(\log_e \lambda) \times \mathbf{F}$, and how can one derive the conclusion of the problem from this?

S2. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (y, z, x)$. Show that there is no integrating factor for $\mathbf{F}$ on the open first octant defined by $x, y, z > 0$.

S3. Let $\mathbf{F} = (P, Q, R)$ be a vector field defined on a region of coordinate 3-space, and define the Laplacian $\nabla^2\mathbf{F}$ by the coordinate-wise formula $(\nabla^2 P, \nabla^2 Q, \nabla^2 R)$. As in the case of scalar functions, we shall say that $\mathbf{F}$ is harmonic if $\nabla^2\mathbf{F} = \mathbf{0}$. Using Exercise $15(a)$, show that $\mathbf{F}$ is harmonic if both $\nabla \times \mathbf{F} = \mathbf{0}$ and $\nabla \cdot \mathbf{F} = 0$.