A surface area computation

PROBLEM. Find the surface area for the graph of the function $f(x, y) = xy$ defined over the disk D of all (x, y) such that $x^2 + y^2 \leq 1$.

The graph of this function is displayed in the following document:

`http://math.ucsd.edu/~gptesler/20c/graphs/saddle.jpg`

Solution. We have

$$\frac{\partial f}{\partial x} \;=\; y\;, \qquad \frac{\partial f}{\partial y} \;=\; x$$

and if we substitute this into the formula for surface area we obtain the following:

$$\text{Surface area} \;=\; \int\int_D \sqrt{1 + x^2 + y^2}\, dA$$

Using polar coordinates we may rewrite this as

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r\, dr\, d\theta \;=\; \pi \int_0^1 \sqrt{1 + r^2} \cdot 2r\, dr$$

and if we make the change of variables $u = r^2$ the latter becomes

$$\pi \int_0^1 \sqrt{1 + u}\, du \;=\; \pi \cdot \left(\frac{2}{3}\right) \cdot (1 + u)^{3/2} \Big|_0^1 \;=\; \frac{2\pi}{3} \left(2\sqrt{2} - 1\right)\;.$$