# Another surface area computation

**PROBLEM.** Set up the integral for the surface area of the parametrized surface $\mathbf{X}(u,v) = (u+v, u^2, v^2)$ where $1 \leq u, v \leq 2$

**Solution.** First of all, we claim that $\mathbf{X}$ is 1–1, for if $\mathbf{X}(u_1,\, v_1) = \mathbf{X}(u_2,\, v_2)$, then $u_1^2 = u_2^2$ and $v_1^2 = v_2^2$, and since $u_i$ and $v_i$ are all positive, it follows that $(u_1,\, v_1) = (u_2,\, v_2)$.

The partial derivatives of $\mathbf{X}$ are given by

$$\frac{\partial \mathbf{X}}{\partial u} \quad = \quad (1, 2u, 0)\ , \qquad \frac{\partial \mathbf{X}}{\partial v} \quad = \quad (1, 2v, 0)$$

and therefore the standard normal vector $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2$ is given by

$$(4uv,\ -2v,\ -2u)$$

so that its length is $\sqrt{16u^2v^2 + 4u^2 + 4v^2}$; since $u$ and $v$ are both positive in the domain of interest, this length is always positive.

Therefore the surface area is given by

$$\int_1^2 \int_1^2 \sqrt{16u^2v^2 + 4u^2 + 4v^2}\, du\, dv\ .$$

The problem does not call for the evaluation of this integral, so we shall not attempt to do so. As noted in the lectures and the file `nonelementary integrals.pdf`, most integrals of this form cannot be expressed in terms of the functions one normally sees in single variable calculus courses.

*Defining equation for the surface.* Here is an easy way, which uses the fact that $u$ and $v$ are positive on the domain of interest in the problem. We can write $u = \sqrt{y}$ and $v = \sqrt{z}$, so that $x = u + v = \sqrt{y} + \sqrt{z}$.