Yet another surface area computation

PROBLEM. Set up the integral for the surface area for the **oblate spheroid** surface with equation

$$x^2 \ + \ y^2 \ + \ \frac{z^2}{k^2} \ = \ 1$$

(where $0 < k < 1$) which can be parametrized by

$$\mathbf{X}(\theta, \phi) \ = \ (\cos\theta \sin\phi, \ \sin\theta\sin\phi, \ k\cos\phi)$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Geometrically, this is the image of the sphere under the linear map sending (x, y, z) to (x, y, kz), so that the line joining the north and south poles is shorter than the line joining two opposite (or antipodal) points on the equator. In fact, the earth has this property so that it is really more like an oblate spheroid than an standard sphere.

Solution. The key point is to compute the length of the normal vector and insert this into the integral formula.

The partial derivatives of the given parametrization $\mathbf{X}$ are equal to

$$\frac{\partial \mathbf{X}}{\partial \theta} \ = \ (-\sin\theta\sin\phi, \ \cos\theta\sin\phi, 0) \ , \qquad \frac{\partial \mathbf{X}}{\partial \phi} \ = \ (\cos\theta\cos\phi, \sin\theta\cos\phi, \ -k\sin\phi)$$

and therefore the standard normal vector $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2$ is given by

$$(-k\sin^2\phi\cos\theta, \ -k\sin^2\phi\sin\theta, \ -\sin\phi\cos\phi)$$

so that its length is equal to

$$\sqrt{k^2\sin^4\phi + \sin^2\phi\cos^2\phi} \ = \ \sin\phi\sqrt{1 + (k^2-1)\sin^2\phi} \ = \ \sin\phi\sqrt{k^2 + (1-k^2)\cos^2\phi} \ .$$

Therefore the surface area of the oblate spheroid is equal to

$$\int_0^{2\pi}\int_0^{\pi} \ \sin\phi\sqrt{k^2 + (1-k^2)\cos^2\phi}\, d\phi\, d\theta \ = \ 2\pi \cdot \int_0^{\pi} \ \sin\phi\sqrt{k^2 + (1-k^2)\cos^2\phi}\, d\phi\, d\theta \ .$$

This integral can be evaluated using the methods of first year calculus; we shall not carry out these computations, but here is a reference for the results:

`http://en.wikipedia.org/wiki/Spheroid`