A surface integral computation for a scalar function

PROBLEM. Set up and evaluate the surface integral

$$\int\int_S z\,dS$$

where S is the upper hemisphere of the unit sphere defined by the equation $x^2+y^2+z^2 = 1$. The "upper" condition corresponds to the inequality $z \geq 0$.

Solution. We shall use the standard spherical coordinate parametrization

$$\mathbf{X}(\theta,\phi) \quad = \quad (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{1}{2}\pi$. As in a previous file, it follows that $|\mathbf{N}(\theta,\phi)| = \sin\phi$, and therefore (since $z = \cos\phi$) the surface integral is given by the following double/iterated integral:

$$\int_0^{2\pi}\int_0^{\pi/2} \cos\phi\ \sin\phi\,d\phi\,d\theta$$

The latter reduces to

$$\pi\int_0^{\pi/2} \sin 2\phi\,d\phi \quad = \quad \pi\ .$$