Flux integral computations

PROBLEM 1. Set up and evaluate the flux integral

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x,y,z) = (x,y,z)$ and $\mathbf{S}$ is the surface $z = 4 - x^2 - y^2$ defined for $x^2 + y^2 \leq 4$, with the **upward** normal orientation.

Solution. This surface is the graph of the function $f(x,y) = 4 - x^2 - y^2$ for $x^2 + y^2 \leq 4$, so we shall use the graph parametrization $\mathbf{X}(x,y) = (x, y, 4 - x^2 - y^2)$.

We shall use the following (very important!) general formula for the upward normal of a surface defined by an equation of the form $z = f(x,y)$:

$$\mathbf{N}(x,y) \quad = \quad (-f_1. - f_2, 1)$$

For the surface of interest here, it follows that

$$\mathbf{N}(x,y) = \big(-(-2x), \, -(-2y), 1\big) \;\; = \;\; (2x, 2y, 1) \; .$$

Since we also have

$$\mathbf{F}\big(\mathbf{X}(x,y)\big) \quad = \quad (x, y, 4 - x^2 - y^2)$$

it follows that the flux $\mathbf{F} \cdot \mathbf{N}$ is given by $2x^2 + 2y^2 + 4 - x^2 - y^2 \;=\; 4 + x^2 + y^2$. Therefore the flux integral is equal to

$$\int\int_D 4 + x^2 + y^2 \, dA$$

where D is the disk defined by $x^2 + y^2 \leq 4$. One simple way to evaluate this integral is by means of polar coordinates, and it follows that the flux integral must be equal to

$$\int_0^{2\pi} \int_0^2 (4 + r^2)\, r \, dr \, d\theta \; .$$

Evaluating this iterated integral from the inside out, we see that it is equal to

$$\int_0^{2\pi} \int_0^2 4r + r^3 \, dr \, d\theta \;\; = \;\; \int_0^{2\pi} 2r^2 + \frac{r^4}{4} \Big|_0^2 \, d\theta \;\; = \;\; \int_0^{2\pi} (8+4)\, d\theta \;\; = \;\; 2\pi \cdot 12 \;\; = \;\; 24\pi \; .$$

PROBLEM 2. (*A modified version of Problem* 15 *on page 438 of the text*). Let S be the boundary of the cylinder defined by $x^2 + y^2 \leq 9$ and $0 \leq z \leq 4$, and take the preferred orientation pointing outward from S. Compute the flux integral of $\mathbf{F}(x, y, z) = (x, y, z+1)$ over S.

Solution. This surface has three separate smooth pieces, and we need to check that each of them is parametrized so that the normal is pointing outward.

(i) Top face of the cylinder. This is given by $x^2 + y^2 \leq 9$ and $z = 4$, so it can be parametrized as the graph of $f(x, y) = 4$. Over this piece of the surface, the outward pointing normal direction is the same as the upward pointing direction, so the given parametrization yields the outward pointing normal, and if we apply the formula from the preceding section we see that $\mathbf{N} = (0, 0, 1)$ on this piece of the surface.

(ii) Lateral face of the cylinder. This is given by $x^2 + y^2 = 9$ and $0 \leq z \leq 4$, so it can be parametrized by $\mathbf{X}(\theta, z) = (3\cos\theta, 3\sin\theta, z)$ for $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 4$. If we compute the normal direction for this parametrization, we obtain $\mathbf{N} = (3\cos\theta, 3\sin\theta, 0)$. This is the outward pointing normal on the lateral face of the surface.

(iii) Bottom face of the cylinder. This is given by $x^2 + y^2 \leq 9$ and $z = 0$, so it can be parametrized as the graph of $f(x, y) = 0$. Over this piece of the surface, the outward pointing normal direction is the **downward** pointing direction, and as in the discussion of the top face the given parametrization defines the upward direction. Therefore we have to adjust the parametrization so that its normal direction points downward. One simple way of reversing the normal direction is to replace the usual parametrization $(x, y, 0)$ with $(x, -y, 0)$, and we choose this parametrization for the bottom part so that its standard normal direction points downward/outward.

To find the flux integral over the entire cylinder, we need to compute the flux integrals for each of the three smooth faces and conclude by adding their values together.

On the top face, we know that $\mathbf{F}\big(\mathbf{X}(x, y)\big) = (x, y, 5)$ and $\mathbf{N} = (0, 0, 1)$, so that $(\mathbf{F} \cdot \mathbf{N}) = 5$. Therefore the flux over the top face is the integral of the constant function 5 over the region $x^2 + y^2 \leq 9$; the value of this integral is equal to 45π (using the formula $A = \pi r^2$ for the area of a solid disk of radius r).

On the lateral face, we have $\mathbf{F}\big(\mathbf{X}(\theta, z)\big) = (3\cos\theta, 3\sin\theta, z)$ and (as noted before) $\mathbf{N} = (3\cos\theta, 3\sin\theta, 0)$, so that $\mathbf{F} \cdot \mathbf{N}$ is equal to 9. Therefore the flux over the top face is the integral of the constant function 9 over the region $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 4$; by the standard formula for the lateral area of a cylinder, the area of the lateral surface is equal to $(2\pi) \times 3 \times 4 = 24\pi$, and therefore the value the integral is equal to $(24\pi) \times 9 = 216\pi$.

Finally, on the bottom face, we know that $\mathbf{F}\big(\mathbf{X}(x,y)\big) \;=\; (x,-y,1)$ and

$$\mathbf{N} \quad = \quad \mathbf{X}_1 \times \mathbf{X}_2 \quad = \quad (1,0,0) \times (0,-1,0) \quad = \quad (0,0,-1)$$

so that $(\mathbf{F} \cdot \mathbf{N}) = -1$. Therefore the flux over the bottom face is the integral of the constant function -1 over the region $x^2 + y^2 \leq 9$; by the same reasoning employed for the top face, we see that the value of this integral is equal to -9π.

COMPUTING THE TOTAL FLUX. Having computed the flux over each of the three pieces (with the correct preferred orientations!), we can find the total flux by adding the values obtained for the three pieces. Therefore the total flux is equal to the sum

$$45\pi \;+\; 216\pi \;-\; 9\pi \quad = \quad 252\pi\ .$$