# Solutions for Sections 5.1 through 5.3

***These solutions use material from the Instructor's Solutions Manual are <u>not</u> to be distributed outside of this class!!***

Section 5.1, p. 291 and following

**2.**

$$\int_0^{\pi}\int_1^2 (y\sin x)\,dy\,dx = \int_0^{\pi}\left(\frac{y^2}{2}\sin x\right)\Bigg|_{y=1}^{y=2} dx = \int_0^{\pi}\left((2\sin x) - \left(\frac{1}{2}\sin x\right)\right) dx$$

$$= \frac{3}{2}\int_0^{\pi}(\sin x)\,dx = -\frac{3}{2}(\cos x)\Big|_0^{\pi} = \frac{3}{2} + \frac{3}{2} = 3.$$

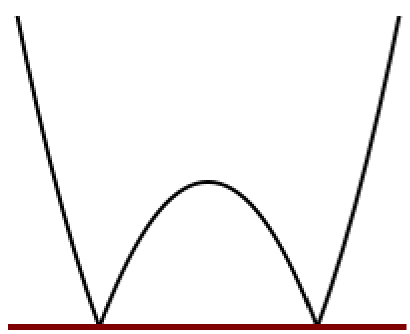
8.

$$\int_1^2\int_0^3 (x + 3y + 1)\,dx\,dy = \int_1^2\left(\frac{x^2}{2} + 3yx + x\right)\Bigg|_0^3 dy = \int_1^2\left(\frac{9}{2} + 9y + 3\right) dy$$

$$= \int_1^2\left(\frac{15}{2} + 9y\right) dy = \left(\frac{15}{2}y + \frac{9}{2}y^2\right)\Bigg|_1^2$$

$$= (15 + 18) - (15/2 + 9/2) = 21.$$

14. Notice that there is an absolute value sign in the integrand. This is always a warning sign that one needs to be careful when doing the computations because sometimes $|x|$ is equal to $x$ and sometimes it is $-x$. Whenever one wants to compute the integral of something like $|f(x)|$ where $f$ is something reasonable, it is necessary to split the integral over the original object into integrals over subintervals on which $f$ is nonnegative and $f$ is nonpositive ; for example, if $f(x) = x^2 - 1$ and $[a, b] = [-2, 2]$, then one must split the latter into three pieces because $f$ is nonnegative on $[-2, -1]$ and $[1, 2]$, while it is nonpositive on $[-1, 1]$.



Graph of the function $f(x) = |x^2 - 1|$

This principle of subdividing the interval appears in the second part of the solution given below.

This is the volume of the region bounded by $z = |x| \sin \pi y$, the $xy$-plane, and the planes $x = -2,\ x = 3,\ y = 0$, and $y = 1$. The volume is

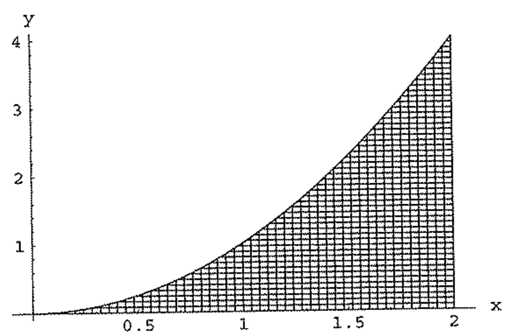
$$V = \int_{-2}^{3} \int_{0}^{1} |x| \sin \pi y \, dy \, dx = \int_{-2}^{3} -\frac{|x|}{\pi} \cos \pi y \Big|_0^1 \, dx = \int_{-2}^{3} \frac{2|x|}{\pi} \, dx.$$

At this point we use the definition of absolute value to split this into two quantities:

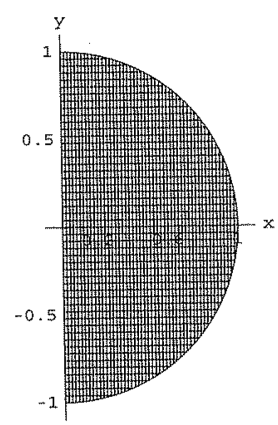
$$V = \int_{-2}^{0} -\frac{2}{\pi} x \, dx + \int_{0}^{3} \frac{2}{\pi} x \, dx = -\frac{x^2}{\pi}\Big|_{-2}^{0} + \frac{x^2}{\pi}\Big|_{0}^{3} = \frac{4}{\pi} + \frac{9}{\pi} = \frac{13}{\pi}.$$

## Section 5.2, p. 307 and following

**4.** $\int_0^2 \int_0^{x^2} y \, dy \, dx = \int_0^2 \frac{y^2}{2}\Big|_0^{x^2} dx = \int_0^2 \frac{x^4}{2} dx = \frac{x^5}{10}\Big|_0^2 = \frac{32}{10} = \frac{16}{5}$. The region over which we are integrating is:

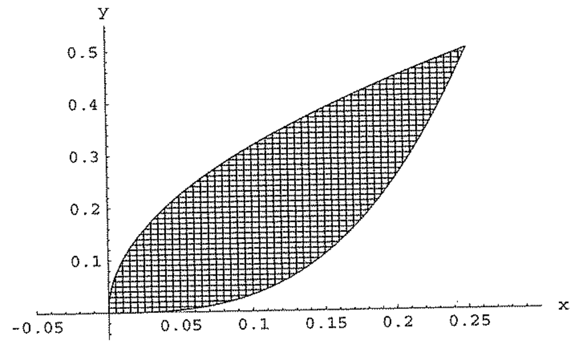



**7.** $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3 \, dy \, dx = \int_0^1 3y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_0^1 6\sqrt{1 - x^2} \, dx =$ (using the substitution $x = \sin t$) $= 3\pi/2$.
You can also see that the region over which we are integrating is a half-circle of radius 1 so we have found the volume of the cylinder over this region of height 3. This figure is:




**8.** This is the same as Exercise 7 with the limits of integration reversed. The solution is again $3\pi/2$.

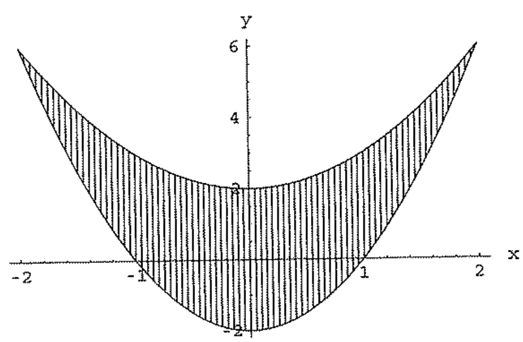
**12.** Again a sketch of the region over which we are integrating helps us set up our double integral. The top bounding curve is $y = \sqrt{x}$ and the bottom curve is $y = 32x^3$.




$$\int_0^{1/4}\int_{32x^3}^{\sqrt{x}} 3xy\,dy\,dx = \int_0^{1/4}\left(\frac{3xy^2}{2}\right)\Bigg|_{32x^3}^{\sqrt{x}} dx = \int_0^{1/4}\left(\frac{3}{2}x^2 - 1536x^7\right)dx$$

$$= \left(\frac{3}{2}x^2 - 192x^8\right)\Bigg|_0^{1/4} = \frac{1}{128} - \frac{3}{1024} = \frac{5}{1024}.$$

15. In the lectures the integral was only set up; here is the complete solution.




$$\iint_D (x - 2y)\,dA = \int_{-2}^{2}\int_{2x^2-2}^{x^2+2}(x - 2y)\,dy\,dx$$

$$= \int_{-2}^{2}(xy - y^2)\Big|_{2x^2-2}^{x^2+2} dx = \int_{-2}^{2}(3x^4 - x^3 - 12x^2 + 4x)\,dx$$

$$= (3x^5/5 - x^4/4 - 4x^3 + 2x^2)\Big|_{-2}^{2} = 192/5 - 64 = -128/5$$

As noted in the lectures, one important point is to recognize which one of the graphs $y = x^2 + 2$, $y = 2x^2 - 2$ lies above the other, and in order to find the limits of integration with respect to $x$ it is necessary to find the points where the two graphs meet. The $x$ – coordinates are given by the solutions to $x^2 + 2 = 2x^2 - 2$.

**24. (a)** For $x \geq 0$ the curve $x^3 - x$ lies below the curve $y = ax^2$ between 0 and their positive point of intersection $x = \dfrac{a + \sqrt{a^2+4}}{2}$. So the area is given by $\displaystyle\int_0^{(a+\sqrt{a^2+4})/2} \int_{x^3-x}^{ax^3} dy\,dx$.

**28. (a)** The function $h(x, y) = f(x)g(y)$ satisfies the conditions of Theorem 2.6 (Fubini's theorem) on $[a, b] \times [c, d]$. So:

$$\iint_R f(x)g(y)\,dA = \int_a^b \int_c^d f(x)g(y)\,dy\,dx.$$

***Note.*** Fubini's Theorem is basically just another name for the Iterated Integral Theorem.

For emphasis, we rewrite this last integral with parentheses and, since $f(x)$ does not depend on $y$, we have:

$$\int_a^b \left( \int_c^d f(x)g(y)\,dy \right) dx = \int_a^b f(x) \left( \int_c^d g(y)\,dy \right) dx.$$

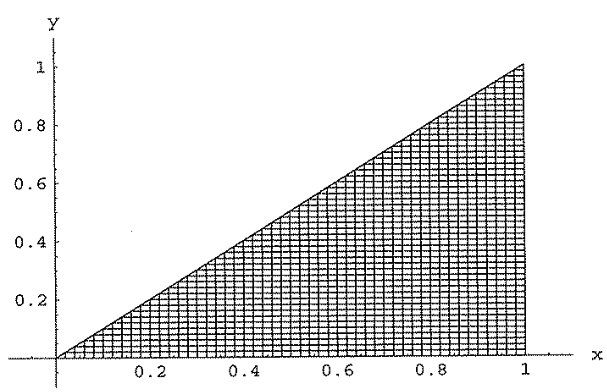
But $\displaystyle\int_c^d g(y)\,dy$ is constant so we can pull it out of this last integral to get the result:

$$\int_a^b f(x) \left( \int_c^d g(y)\,dy \right) dx = \left( \int_c^d g(y)\,dy \right) \left( \int_a^b f(x)\,dx \right).$$

Another formula of this type is given in the additional exercises for Section 5.3.

## Section 5.3, p. 311 and following

**2.** The region of integration is:



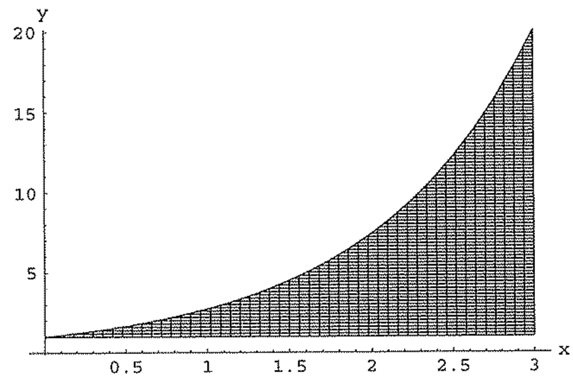

$$\int_0^1 \int_0^x (2 - x - y)\,dy\,dx = \int_0^1 (2x - 3x^2/2)\,dx = 1/2 \quad \text{and}$$
$$\int_0^1 \int_y^1 (2 - x - y)\,dx\,dy = \int_0^1 \frac{3}{2}(y^2 - 2y + 1)\,dy = 1/2.$$

3. As shown in the lectures, each of the iterated integrals below is equal to the double integral over the triangular region in the first quadrant bounded by the coordinate axes and the

graph of $y = 4 - 2x$; since no illustration is included, we shall note that the vertices of the boundary triangle are the origin, the point **(0, 4),** and the point **(2, 0).** Here are the computations in both orders which yield the common value.

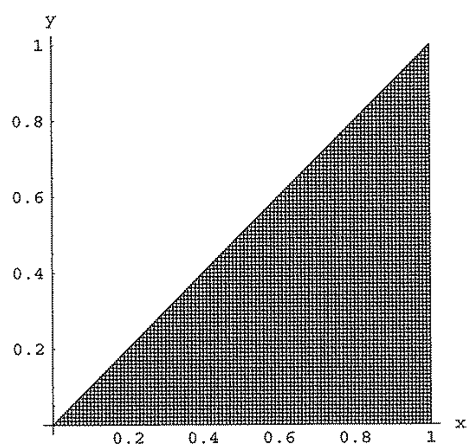
$$\int_0^2 \int_0^{4-2x} y\,dy\,dx = \int_0^2 (2x^2 - 8x + 8)\,dx = 16/3 \quad \text{and}$$
$$\int_0^4 \int_0^{2-y/2} y\,dx\,dy = \int_0^4 (-y^2/2 + 2y)\,dy = 16/3.$$

**6.** The region of integration is:

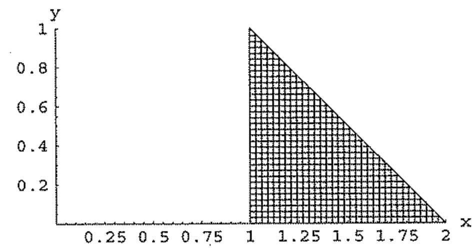



$$\int_0^3 \int_1^{e^x} 2\,dy\,dx = \int_0^3 (2e^x - 2)\,dx = 2e^3 - 8 \quad \text{and}$$
$$\int_1^{e^3} \int_{\ln y}^3 2\,dx\,dy = \int_0^3 (6 - 2\ln y)\,dy = 2e^3 - 8.$$

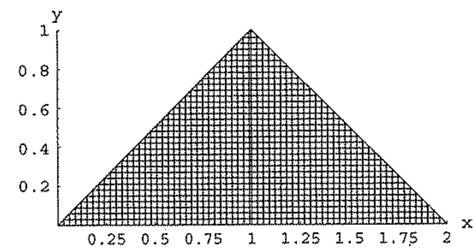
**12.** The limits of integration of the first integral describe the triangular region $D_1$ bounded on top by $y = x$:



The limits of integration of the second integral describe the triangular region $D_2$ bounded by $y = 2 - x$:




Taken together, we obtain the triangular region $D$ below




Reversing the order of integration, we find that the sum of the integrals equals

$$\int_0^1 \int_y^{2-y} \sin x \, dx \, dy = \int_0^1 (-\cos(2 - y) + \cos y) \, dy$$

$$= (\sin(2 - y) + \sin y)\Big|_0^1 = \sin 1 + \sin 1 - \sin 2$$

$$= 2 \sin 1 - \sin 2.$$

This and similar exercises illustrate how interchanging the order of integration can be used to simplify computations.

**14.** We reverse the order of integration:

$$\int_0^1 \int_{3y}^3 \cos x^2 \, dx \, dy = \int_0^3 \int_0^{x/3} \cos x^2 \, dy \, dx = \int_0^3 (y \cos x^2)\Big|_0^{x/3} dx$$

$$= \frac{1}{3} \int_0^3 x \cos x^2 \, dx = \frac{\sin x^2}{6}\Big|_0^3 = \frac{\sin 9}{6}.$$

***<u>Note.</u>*** In a one – hour examination, it is unlikely that there will be a problem which requires both interchanging the order of integration and evaluating the same integral.