Solutions for Sections 5.4 through 5.8

These solutions use material from the Instructor's Solutions Manual are <u>not</u> to be distributed outside of this class!!

Section 5.4, p. 321 and following

2. Here order doesn't matter.

$$\iiint_{[0,1]\times[0,2]\times[0,3]} (x^2 + y^2 + z^2)\,dV = \int_0^1 \int_0^2 \int_0^3 (x^2 + y^2 + z^2)\,dz\,dy\,dx$$
$$= \int_0^1 \int_0^2 \left(x^2 z + y^2 z + \frac{z^3}{3} \right)\Big|_0^3 dy\,dx$$
$$= \int_0^1 \int_0^2 (3x^2 + 3y^2 + 9)\,dy\,dx$$
$$= \int_0^1 (3x^2 y + y^3 + 9y)\Big|_0^2 dx$$
$$= \int_0^1 (6x^2 + 26)\,dx$$
$$= (2x^3 + 26x)\Big|_0^1 = 28.$$

4. This works for the same reason that Exercise 1 simplified. We are integrating an odd function of z on an interval that is symmetric in the z coordinate and so, since $\int_{-3}^{3} z\,dz = 0$, the triple integral will also be 0.

5.

$$\int_{-1}^{2} \int_{1}^{z^2} \int_0^{y+z} 3yz^2\,dx\,dy\,dz = \int_{-1}^{2} \int_{1}^{z^2} 3xyz^2 \Big|_0^{y+z} dy\,dz = 3\int_{-1}^{2} \int_{1}^{z^2} (y^2 z^2 + yz^3)\,dy\,dz$$
$$= 3\int_{-1}^{2} \left(\frac{y^3 z^2}{3} + \frac{y^2 z^3}{2} \right)\Big|_1^{z^2} dz = 3\int_{-1}^{2} \left(\frac{z^8}{3} + \frac{z^7}{2} - \frac{z^3}{2} - \frac{z^2}{3} \right) dz$$
$$= 3\left(\frac{z^9}{27} + \frac{z^8}{16} - \frac{z^4}{8} - \frac{z^3}{9} \right)\Big|_{-1}^{2} = \frac{1539}{16}.$$

8.

(b) Work out that the equation of the circle where the two paraboloids intersect is $x^2 + y^2 = 9/2$ so

$$\text{Volume} = \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}} \int_{x^2+y^2}^{9-x^2-y^2} 1\,dz\,dy\,dx$$
$$= \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}} (9 - 2x^2 - 2y^2)\,dy\,dx$$

$$= \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \left(\left[12 - \frac{8}{3}x^2 \right] \sqrt{\frac{9}{2} - x^2} \right) dx$$

$$= \left(\sqrt{\frac{9}{2} - x^2} \left[\frac{15x}{2} - \frac{2x^3}{3} \right] + \frac{81}{4} \arcsin \left[\frac{\sqrt{2}x}{3} \right] \right) \Bigg|_{-3/\sqrt{2}}^{3/\sqrt{2}}$$

$$= \frac{81\pi}{4}.$$

20.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (2\sqrt{a^2 - x^2})\,dy\,dx$$

$$= \int_{-a}^{a} 4(a^2 - x^2)\,dx$$

$$= \frac{16a^3}{3}.$$

Section 5.5, p. 307 and following

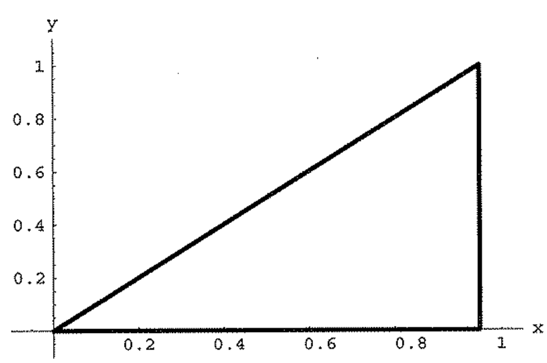
4. We are trying to determine the entries a, b, c, and d in the expression:

$$\mathbf{T}(u, v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since $\mathbf{T}(0,0) = (0,0)$ we know that the motion is not a translation. Also $\mathbf{T}(0,5) = (4,1)$ so $b = 4/5$ and $d = 1/5$. Now $\mathbf{T}(1,2) = (1,-1)$ so $a = -3/5$ and $c = -7/5$. We check with the remaining vertex: $\mathbf{T}(-1,3) = (3,2)$.

$$\mathbf{T}(u, v) = \begin{bmatrix} -3/5 & 4/5 \\ -7/5 & 1/5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

6. You can see that $\mathbf{T}(u, v) = (u, uv)$ is not one-one on D^* by observing that all points of the form $(0, v)$ get mapped to the origin under $\mathbf{T}$. In fact, you can imagine the map by picturing the left vertical side of the unit square being shrunk down to a point at the origin. The image is the triangle:



8. **(a)** $\displaystyle\int_0^1 \int_{y/2}^{(y/2)+2} (2x - y)\,dx\,dy = \int_0^1 (x^2 - xy)\Big|_{y/2}^{(y/2)+2} dy = \int_0^1 4\,dy = 4.$ A sketch of D is shown below.

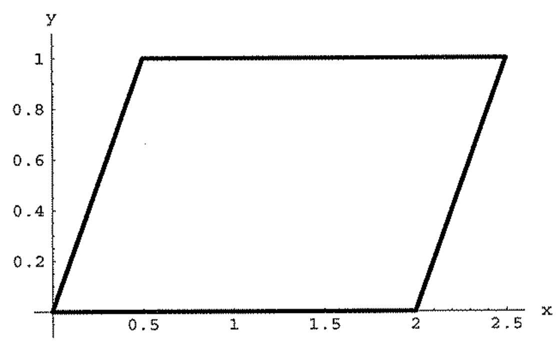
(b) We again can apply Proposition 5.1 and see that the vertices are mapped: $(0,0) \to (0,0), (2,0) \to (4,0), (1/2,1) \to (0,1)$, and $(5/2,1) \to (4,1)$ so D^* is $[0,4] \times [0,1]$.

(c) First note that

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Then, using the change of variables theorem,

$$\int_0^1 \int_{y/2}^{(y/2)+2} (2x - y)\,dx\,dy = \int_0^1 \int_0^4 u(1/2)\,du\,dv = \int_0^1 \frac{u^2}{4}\Big|_0^4 du = \int_0^1 4\,dv = 4.$$



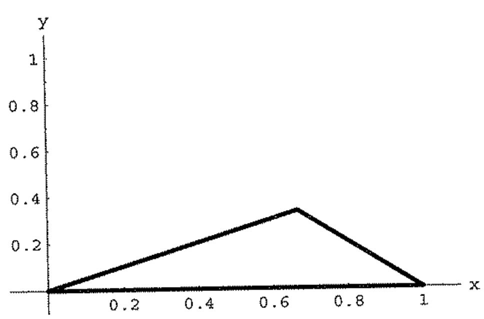
9. First,

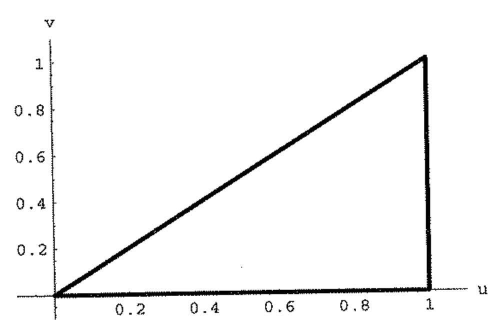
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Also we can rewrite $x^5(2y - x)e^{(2y-x)^2} = u^5 v e^{v^2}$, and the transformed region is $[0,2] \times [0,2]$ so

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2x - y)e^{(2x-y)^2}\,dy\,dx = \int_0^2 \int_0^2 u^5 v e^{v^2}(1/2)\,du\,dv = \frac{16}{3}\int_0^2 v e^{v^2}\,dv = \frac{8}{3}(e^4 - 1).$$

10. The original region D is sketched below left. The transformation $u = x + y$ and $v = x - 2y$ maps D to the region D^* sketched below right. $\partial(x,y)/\partial(u,v)$ is easily calculated to be $-1/3$.





So, using the change of variables theorem, our integral becomes

$$\int_0^1 \int_0^u \frac{1}{3}\left(\frac{u^{1/2}}{v^{1/2}}\right) dv\,du = \int_0^1 \frac{2}{3}u^{1/2}v^{1/2}\Big|_0^u du = \int_0^1 \frac{2}{3}u\,du = \frac{u^2}{3}\Big|_0^1 = \frac{1}{3}.$$

11. Here the problem cries out to you to let $u = 2x + y$ and $v = x - y$. Once you've made that move you can easily figure that $\partial(x,y)/\partial(u,v) = -1/3$ and that the new region is $[1,4] \times [-1,1]$. So the integral is

$$\int_1^4 \int_{-1}^1 u^2 e^v (1/3)\, dv\, du = \frac{1}{3}\int_1^4 u^2 e^v \Big|_{-1}^1 du = (e - e^{-1})\frac{u^3}{9}\Big|_1^4 = 7(e - e^{-1}).$$

The preceding solution suppresses an important step: namely, writing down explicit solutions for $\boldsymbol{x}$ and $\boldsymbol{y}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$. These solutions can be formed by the usual methods for solving two linear equations in two unknowns, and the answers are $\boldsymbol{x = (u + v)/3,\ y = (u - 2v)/3.}$

16. $$\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^a r e^{r^2}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}e^{r^2}\Big|_0^a d\theta = \int_{-\pi/2}^{\pi/2}\frac{1}{2}(e^{a^2}-1)d\theta = \pi(e^{a^2}-1)/2.$$

24. Since B is a ball we will use spherical coordinates:

$$\begin{aligned}
\iiint_B \frac{dV}{\sqrt{x^2+y^2+z^2+3}} &= \int_0^{2\pi}\int_0^{\pi}\int_0^2 \frac{\rho^2 \sin\varphi}{\sqrt{\rho^2+3}}\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\left(\left[\sqrt{7} - \frac{3}{2}\operatorname{arcsinh}(2/\sqrt{3})\right]\sin\varphi\right)d\varphi\,d\theta \\
&= \int_0^{2\pi}(2\sqrt{7} - 3\operatorname{arcsinh}(2/\sqrt{3}))\,d\theta \\
&= 4\sqrt{7}\pi - 6\pi\operatorname{arcsinh}(2/\sqrt{3}) \quad \text{which is the same as the text's solution} \\
&= (4\sqrt{7} - 6\ln(2+\sqrt{7}) + 3\ln 3)\pi.
\end{aligned}$$

28. We are integrating over a cone with vertex at the origin and base the disk at height 6 with radius 3. We will use cylindrical coordinates.

$$\begin{aligned}
\iiint_W (2 + \sqrt{x^2+y^2})\,dV &= \int_0^3\int_0^{2\pi}\int_{2r}^{6} r(2+r)\,dz\,d\theta\,dr \\
&= \int_0^3\int_0^{2\pi}(-2r^3 + 2r^2 + 12r)\,d\theta\,dr \\
&= \int_0^3 (2\pi(-2r^3+2r^2+12r))\,dr = 63\pi.
\end{aligned}$$

Section 5.6, p. 355 and following

For these problems, it is enough to set up the integrals without evaluating them explicitly. However, we include the entire solutions for the sake of completeness.

10. (a) The curve $y = 8 - 2x^2$ intersects the x-axis at ± 2. So

$$\int_{-2}^{2}\int_{0}^{8-2x^2} c\,dy\,dx = c\int_{-2}^{2}(8 - 2x^2)\,dx = c\,(8x - 2x^3/3)\Big|_{-2}^{2} = 64c/3$$

$$M_y = \int_{-2}^{2}\int_{0}^{8-2x^2} cx\,dy\,dx = c\int_{-2}^{2}(8x - 2x^3)\,dx = c\,(4x^2 - x^4/2)\Big|_{-2}^{2} = 0 \quad \text{and}$$

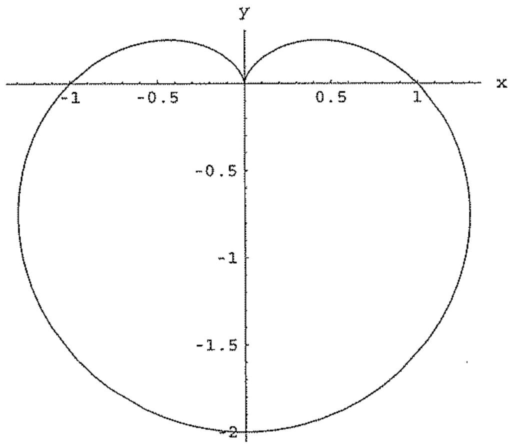
$$M_x = \int_{-2}^{2}\int_{0}^{8-2x^2} cy\,dy\,dx = (c/2)\int_{-2}^{2}(8 - 2x^2)^2dx = c\left(\frac{2}{5}x^5 - \frac{16}{3}x^3 + 32x\right)\Big|_{-2}^{2} = 1024c/15$$

So $\overline{x} = 0$ and $\overline{y} = \dfrac{1024c/15}{64c/3} = 16/5$.

(b) Again, we see the symmetry with respect to x so $\overline{x} = 0$. The following integrals are straightforward so we leave out the details, but

$$\overline{y} = \frac{\displaystyle\int_{-2}^{2}\int_{0}^{8-2x^2} 3cy^2\,dy\,dx}{\displaystyle\int_{-2}^{2}\int_{0}^{8-2x^2} 3cy\,dy\,dx} = \frac{32768c/35}{1024c/5} = 32/7.$$

14. We'll take δ to be 1. A look at the figure below tells us again that $\overline{x} = 0$. We'll use polar integrals to calculate $\overline{y}$.



We first calculate $M = \displaystyle\int_{0}^{2\pi}\int_{0}^{1-\sin\theta} r\,dr\,d\theta = \frac{3\pi}{2}$ and $M_x = \displaystyle\iint_D y\,dA = \int_{0}^{2\pi}\int_{0}^{1-\sin\theta} r^2\sin\theta\,dr\,d\theta = \frac{-5\pi}{4}$, so $\overline{y} = \dfrac{-5\pi/4}{3\pi/2} = -5/6$.

Section 5.8, p. 357 and following

This is worked out at the end of the file **http://math.ucr.edu/~res/math10B/comments05.pdf**.