Solutions for Sections 6.1 through 6.5

These solutions use material from the Instructor's Solutions Manual are <u>not</u> to be distributed outside of this class!!

Section 6.1, p. 379 and following

2.

$$\int_{\mathbf{x}} f\,ds = \int_0^2 [(t)(2t)(3t)\sqrt{1^2 + 2^2 + 3^2}]\,dt = 6\sqrt{14}\int_0^2 t^3\,dt = \frac{6\sqrt{14}}{4}t^4\Big|_0^2 = 24\sqrt{14}.$$

6.

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 (2t+1, t, 3t-1)\cdot(2,1,3)\,dt = \int_0^1 (14t-1)\,dt = (7t^2 - t)\Big|_0^1 = 6.$$

7.

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{\pi/2} (2-\cos t, \sin t)\cdot(\cos t, \sin t)\,dt = \int_0^{\pi/2} (2\cos t - \cos^2 t + \sin^2 t)\,dt$$

$$= (2\sin t - (\sin 2t)/2)\Big|_0^{\pi/2} = 2.$$

12. $$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^2 (2t, 1, 0)\cdot(1, 6t, 0)\,dt = \int_0^2 8t\,dt = 16.$$

20. Parametrize C as $\mathbf{x}(t) = (t^2, t^3)$, $-1 \le t \le 1$, so that $\mathbf{x}'(t) = (2t, 3t^2)$. Then

$$\int_C x^2y\,dx - xy\,dy = \int_{-1}^1 (t^7(2t) - t^5(3t^2))\,dt = \int_{-1}^1 (2t^8 - 3t^7)\,dt = \left(\frac{2}{9}t^9 - \frac{3}{8}t^8\right)\Big|_{-1}^1 = \frac{4}{9}.$$

Section 6.2, p. 389 and following

4. $M(x,y) = 2y$ and $N(x,y) = x$.

- For the line integral, the path is split into two pieces: $\mathbf{x}_1(t) = (a\cos t, a\sin t)$, $0 \le t \le \pi$, and $\mathbf{x}_2(t) = (-a + 2at, 0)$, $0 \le t \le 1$.

$$\oint_{\partial D} M\,dx + N\,dy = \int_0^\pi (2a\sin t, a\cos t)\cdot(-a\sin t, a\cos t)\,dt + \int_0^1 a(0)\,dt$$

$$= a^2\int_0^\pi (-2\sin^2 t + \cos^2 t)\,dt = a^2\int_0^\pi (-2 + 3\cos^2 t)\,dt = -\frac{\pi a^2}{2}.$$

- We'll use polar coordinates for the area calculation:

$$\iint_D (N_x - M_y)\,dA = \iint_D (1 - 2)\,dA = \int_0^{\pi}\int_0^{a} -r\,dr\,d\theta = \int_0^{\pi} -\frac{a^2}{2}\,d\theta = -\frac{\pi a^2}{2}.$$

7. Green's Theorem relates the double integral over a region D with the line integral of the boundary curve Γ where the latter curve is parametrized in the counterclockwise sense. If we parametrize this curve in the clockwise sense, the effect is to change the sign of the line integral, and therefore the double integral is equal to the negative of the line integral for Γ if one takes a parametrization in the counterclockwise sense. Therefore we have the following:

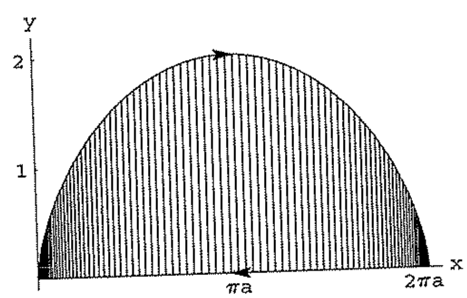
$$\oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy = -\int_0^1\int_0^1 (2x + 2y)\,dy\,dx = -\int_0^1 (2x + 1)\,dx = -2.$$

8. As we saw in Section 6.1, Work $= \oint_C \mathbf{F} \cdot d\mathbf{s}$. If D is the ellipse $x^2 + 4y^2 = 4$ and its boundary is C, then by Green's theorem

$$\oint_C (4y - 3x, x - 4y) = \iint_D (1 - 4)\,dA = \int_{-2}^{2}\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} -3\,dy\,dx$$

$$= \int_{-2}^{2}\left[-6\sqrt{1 - x^2/4}\right]dx = -6\pi.$$

10. One arch of the cycloid is produced from $t = 0$ to $t = 2\pi$.



Because of the orientation shown,

$$\text{Area} = \frac{1}{2}\oint_C y\,dx - x\,dy = \frac{1}{2}\int_{C_1} + \frac{1}{2}\int_{C_2}.$$

Now we use the line integrals to compute the area. The second line integral is trivial to evaluate, for the path in question is a horizontal line segment with $y = 0$, so that the integrand is equal to $0\,dx + x\,d0 = 0$. Therefore the area equals half the line integral along the first curve.

$$\frac{1}{2}\int_{C_1} = \frac{1}{2}\int_0^{2\pi} (a(1 - \cos t) \cdot a(1 - \cos t) - a(t - \sin t) \cdot a \sin t)\,dt$$
$$= \frac{a^2}{2}\int_0^{2\pi} ((1 - \cos t)^2 - t\sin t + \sin^2 t)\,dt = \frac{a^2}{2}\int_0^{2\pi} (1 - 2\cos t + \cos^2 t - t\sin t + \sin^2 t)\,dt$$
$$= \frac{a^2}{2}\int_0^{2\pi} (2 - 2\cos t - t\sin t)\,dt = \frac{a^2}{2}(2t - 2\sin t + t\cos t - \sin t)\Big|_0^{2\pi} = \frac{a^2}{2}(4\pi + 2\pi) = 3\pi a^2.$$

Section 6.3, p. 399 and following

3. $\dfrac{\partial N}{\partial x} = ye^{xy} \neq e^{x+y} = \dfrac{\partial M}{\partial y}$, so **F** is not conservative.

4. $\dfrac{\partial N}{\partial x} = 2x\cos y = \dfrac{\partial M}{\partial y}$, so **F** is conservative. We want to find f where $\mathbf{F} = \nabla f(x, y)$. We find that the indefinite integral of $2x \sin y$ with respect to x is $x^2 \sin y$. To see whether any adjustments need to be made, we check to make certain that $\dfrac{\partial}{\partial y}(x^2 \sin y) = x^2 \cos y$. It does, so we conclude that $f(x, y) = \nabla(x^2 \sin y)$.

7. Note that $\dfrac{\partial}{\partial y}(e^{-y} - y\sin(xy)) = -e^{-y} - \sin xy - xy\cos xy = \dfrac{\partial}{\partial x}(-xe^{-y} - x\sin xy)$. Since the domain of **F** is all of $\mathbf{R}^2$, the vector field is conservative. Thus $\mathbf{F} = \nabla f$, so $\dfrac{\partial f}{\partial x} = e^{-y} - y\sin xy \Rightarrow f(x, y) = xe^{-y} + \cos xy + g(y)$ for some g. Hence $\dfrac{\partial f}{\partial y} = -xe^{-y} - x\sin xy + g'(y) = -xe^{-y} - x\sin xy$ so $g'(y) = 0$. Thus $f(x, y) = xe^{-y} + \cos xy + C$ is a potential for any C.

14. We see that, for **G**, $\partial N/\partial x = 2x = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 2y = \partial N/\partial z$. So $\mathbf{G} = (2xy, x^2 + 2yz, y^2)$ is conservative and $\mathbf{G} = \nabla(x^2y + y^2z)$. We know, therefore, that **F** is not conservative because the wording of the problem assured us that exactly one of **F** and **G** was conservative. It may be more satisfying to verify that for **F**, $\partial M/\partial y = 2xyz^3$ while $\partial N/\partial x = 4xy$. These are different so **F** is not conservative.

Section 6.4, p. 400 and following

7. **TRUE.** This can be verified using Green's Theorem.

Section 6.5, p. 401 and following

36. **(a)**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} & 0 \end{vmatrix} = (0, 0, 0).$$

(b) Here the path is $\mathbf{x}(\theta) = (\cos\theta, \sin\theta)$ for $0 \le \theta \le 2\pi$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} ((-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta))\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

(c) We saw in part (b) that the line integral around a closed path is not zero, so **F** cannot be conservative on its domain.

(d) The conditions are not met for the theorem as the domain of **F** is not a simply-connected region.