Solutions for Sections 7.1 through 7.5

These solutions use material from the Instructor's Solutions Manual are <u>not</u> to be distributed outside of this class!!

Section 7.1, p. 417 and following

1. (a) To find a normal vector we calculate

$$\mathbf{T}_s(s,t) = (2s, 1, 2s) \quad \text{so} \quad \mathbf{T}_s(2,-1) = (4,1,4)$$
$$\mathbf{T}_t(s,t) = (-2t, 1, 3) \quad \text{so} \quad \mathbf{T}_t(2,-1) = (2,1,3).$$

Then a normal vector is

$$\mathbf{N}(2,-1) = \mathbf{T}_s(2,-1) \times \mathbf{T}_t(2,-1) = (-1,-4,2).$$

(b) We find an equation for the tangent plane using

$$0 = \mathbf{N}(2,-1)\cdot(\mathbf{x} - (3,1,1)) = (-1,-4,2)\cdot(\mathbf{x} - (3,1,1)) = -x + 3 - 4y + 4 + 2z - 2.$$

This is equivalent to $x + 4y - 2z = 5$.

<u>Addendum to Exercise 7.1.1.</u> Find an equation in x, y and z which is satisfied by points on the surface.

We have three equations $x = u^2 - v^2$, $y = u + v$, $z = u^2 + 3v$ in the five variables x, y, z, u, v; the objective is to derive one nontrivial equation in x, y, z by eliminating the two variables u and v.

More generally, given p equations in q unknowns where $p < q$, the idea is to start by using one equation to eliminate one variable, obtaining a system of $p - 1$ equations in $q - 1$ unknowns, and to continue eliminating variables successively until we obtain a single equation in $q - p + 1$ unknowns.

For the sake of convenience, let's assume that $y \neq 0$. In this case we know that $w = x/y$ is equal to $u + v$. Thus we can solve for u and v in terms of x and y, obtaining the equations

$$u = \tfrac{1}{2}(y + w) \qquad v = \tfrac{1}{2}(y - w).$$

We can now substitute these values into the formula for z to obtain an expression for z in terms of x and y (strictly speaking, we first obtain a formula in terms of w and y, and then we can substitute for w to get a formula in terms of x and y). Here is the formula for z:

$$\frac{x^2}{4y^2} + \frac{x}{2} + \frac{y^2}{4} + \frac{3y}{2} - \frac{3x}{2y}$$

If we clear the fractions from this equation by multiplying both sides by $4y^2$, here is what we obtain:

$$4y^2z = x^2 + 2xy^2 + y^4 + 6y^3 - 6xy$$

Strictly speaking, we have not addressed the question, "What happens if $y = 0$?" Usually it is all right to ignore such points, but for the sake of completeness we indicate one way to address this issue. — If we define a new variable $w = u + v$, then we may write $x = yw$ and obtain a system of three equations in w, y, z, u, v. Then we can solve for u and v in terms of y and w exactly as before, and using these solutions we can write z in terms of y and w as follows:

$$z = \frac{w^2}{4} + \frac{yw}{2} + \frac{y^2}{4} + \frac{3y}{2} - \frac{3w}{2}$$

If we multiply both sides of this by $4y^2$ and use the identity $x = yw$, then we obtain the previous equation for $4y^2z$ in terms of y and x.

2. First we figure that since $2\sin t = 1$, either $t = \pi/6$ or $5\pi/6$. Since $2\cos t < 0$ we know that $t = 5\pi/6$. Then we can see that $\sin s = \sqrt{2}/2$ so $s = \pi/4$. Next, find a normal vector to the surface at the given point by calculating

$$\mathbf{T}_s(s,t) = (-(5 + 2\cos t)\sin s, (5 + 2\cos t)\cos s, 0) \quad \text{and}$$
$$\mathbf{T}_t(s,t) = (-2\sin t\cos s, -2\sin t\sin s, 2\cos t) \quad \text{so}$$
$$\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = 2(5 + 2\cos t)(\cos s\cos t, \sin s\cos t, \sin t). \quad \text{Therefore,}$$
$$\mathbf{N}(\pi/4, 5\pi/6) = \frac{\sqrt{3} - 5}{\sqrt{2}}(\sqrt{3}, \sqrt{3}, -\sqrt{2}).$$

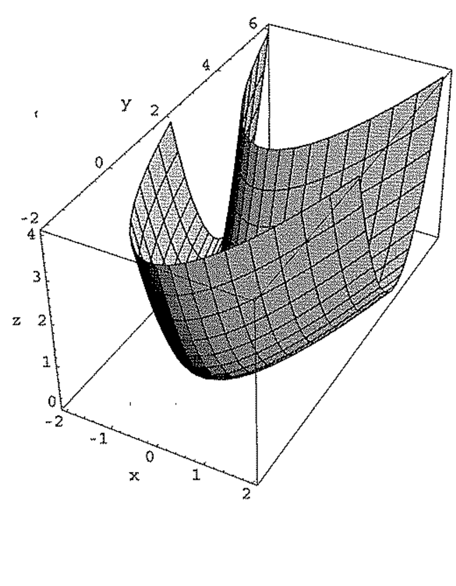
We calculate an equation for the tangent plane by writing $\mathbf{N} \cdot (\mathbf{x} - (x_0, y_0, z_0)) = 0$ or, equivalently in this case,

$$0 = (\sqrt{3}, \sqrt{3}, -\sqrt{2}) \cdot \left(\mathbf{x} - \left(\frac{5 - \sqrt{3}}{\sqrt{2}}, \frac{5 - \sqrt{3}}{\sqrt{2}}, 1\right)\right) \quad \text{or} \quad \sqrt{3}x + \sqrt{3}y - \sqrt{2}z = 5\sqrt{6} - 4\sqrt{2}.$$

4.

(c) Note that the x-component of $\mathbf{X}$ is $s^2\cos t$ and the y-component is $s^2\sin t$ and the z-component is a function of s. We can eliminate the t by looking at $x^2 + y^2$. So without much work we have found that an equation for the image of $\mathbf{X}$ is $x^2 + y^2 - z^4 = 0$.

5.(a) Here is a sketch of the surface:



(b) To determine whether the surface is smooth we need to calculate $\mathbf{N}$. First, $\mathbf{T}_s(s,t) = (1, 2s, 0)$, and $\mathbf{T}_t(s,t) = (0, 1, 2t)$ so $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (4st, -2t, 1)$. We conclude that $\mathbf{N} \neq \mathbf{0}$ for any (s, t) so $\mathbf{N}$ is smooth.

(c) If $(s, s^2 + t, t^2) = (1, 0, 1)$, then $s = 1$ and $t = -1$. So $\mathbf{N}(1, -1) = (-4, 2, 1)$ and an equation of the tangent plane at this point is $(-4, 2, 1) \cdot (\mathbf{x} - (1, 0, 1)) = 0$ or more simply, $4x - 2y - z = 3$.

<u>Addendum to Exercise 7.1.5.</u> Find an equation in x, y and z which is satisfied by points on the surface.

Here we have the three equations $x = u,\ y = u^2 + v,\ z = v^2$ in the five variables. We may use the substitution in the first equation to conclude that $y = x^2 + v$ or equivalently $y - x^2 = v$. Finally, we may substitute this into the third equation to conclude that

$$z = (y - x^2)^2.$$

12. (a) $\mathbf{X}(1, -1) = (1, -1, -1)$ and we have $\mathbf{T}_s = (3s^2, 0, t), \mathbf{T}_t = (0, 3t^2, s)$. Hence the normal at $(1, -1, -1)$—which is when $s = 1, t = -1$—is $\mathbf{N}(1, -1) = \mathbf{T}_s(1, -1) \times \mathbf{T}_t(1, -1) = (3, 0, -1) \times (0, 3, 1) = (3, -3, 9)$. So an equation for the tangent plane is

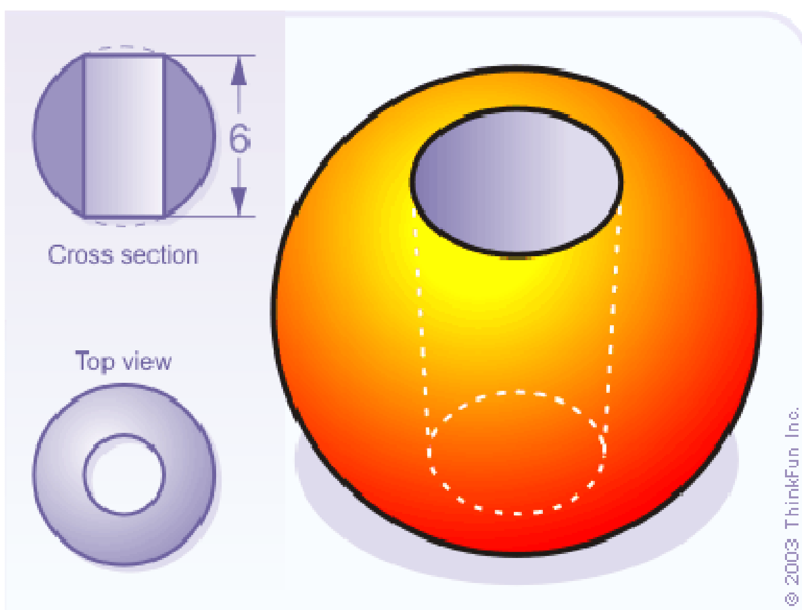
$$3(x - 1) - 3(y + 1) + 9(z + 1) = 0 \quad \text{or} \quad x - y + 3z = -1.$$

(b) In general we have that the standard normal is given by

$$\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3s^2 & 0 & t \\ 0 & 3t^2 & s \end{vmatrix} = (-3t^3, -3s^3, 9s^2t^2).$$

Note that $\mathbf{N} = \mathbf{0}$ when $s = t = 0$, i.e., at $(0, 0, 0)$. So the surface fails to be smooth there.

21. Here are some drawings that might be helpful:

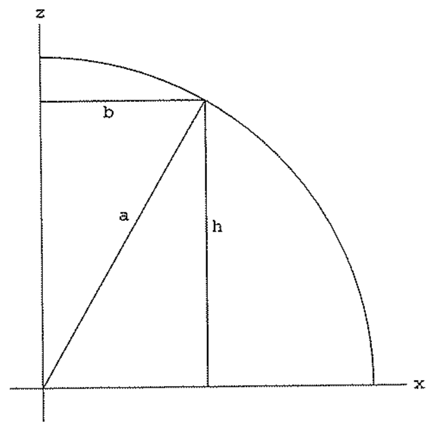


(Source: http://www.puzzles.com/puzzleplayground/HoleInTheSphere/HoleInTheSphere.htm)

[Disregard the number 6 in the cross – sectional view.]

Here is the solution with a more "mathematical" drawing.

21. A quick look at the figure below shows a cutaway of a quarter of the xz-plane intersection of the cylindrical hole of radius b bored in a sphere of radius a. The height of the hole is $2\sqrt{a^2 - b^2}$. The top half of the ring is the region swept out by the portion of the diagram containing the letter 'h'.



If $\mathbf{X}(s,t) = (a\sin s\cos t, a\sin s\sin t, a\cos s)$, then $\mathbf{T}_s(s,t) = (a\cos s\cos t, a\cos s\sin t, -a\sin s)$, $\mathbf{T}_t(s,t) = (-a\sin s\sin t, a\sin s\cos t, 0)$, $\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = a^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)$ and $\|\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t)\| = a^2\sin s$. Notice that the angle s made with the z-axis has lower limit $\cos^{-1}(h/a) = \cos^{-1}(\sqrt{a^2 - b^2}/a)$ and upper limit $\pi/2$. So the surface area is

$$2\int_0^{2\pi}\int_{\cos^{-1}(\sqrt{a^2-b^2}/a)}^{\pi/2} a^2\sin s\, ds\, dt = 2\int_0^{2\pi} a^2\left(\frac{\sqrt{a^2 - b^2}}{a}\right) dt = 4\pi a\sqrt{a^2 - b^2}.$$

22. The parametrization of the paraboloid is $\mathbf{X}(s,t) = (s\cos t, s\sin t, 9 - s^2)$ where $0 \le t \le 2\pi$ and $0 \le s \le 3$. So $\mathbf{T}_s(s,t) = (\cos t, \sin t, -2s)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (2s^2\cos t, 2s^2\sin t, s)$, and $\|\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t)\| = s\sqrt{4s^2 + 1}$. The surface area is then

$$\int_0^{2\pi}\int_0^3 s\sqrt{4s^2 + 1}\, ds\, dt = \frac{1}{12}\int_0^{2\pi} [(1 + 4s^2)^{3/2}]\Big|_0^3\, dt = \frac{\pi}{6}(37^{3/2} - 1).$$

Section 7.2, p. 438 and following

2. (a) Since $\mathbf{X}(s,t) = (s + t, s - t, st)$, we can calculate $\mathbf{T}_s(s,t) = (1, 1, t)$, $\mathbf{T}_t(s,t) = (1, -1, s)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (s + t, t - s, -2)$, and $\|\mathbf{N}(s,t)\| = \sqrt{2s^2 + 2t^2 + 4}$. Using polar coordinates in the double integral, we obtain

$$\iint_{\mathbf{X}} 4\, dS = \iint_D 4\sqrt{2s^2 + 2t^2 + 4}\, ds\, dt = \int_0^{\pi/2}\int_0^1 4r\sqrt{2r^2 + 4}\, dr\, d\theta$$
$$= \frac{2}{3}\int_0^{\pi/2} [6\sqrt{6} - 8]\, d\theta = \frac{\pi}{3}[6\sqrt{6} - 8].$$

(b) By Definition 2.2, $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt$. Here $\mathbf{F}(\mathbf{X}(s,t)) = (s + t, s - t, st)$ and so, from part (a), we know that $\mathbf{N}(s,t) = (s + t, t - s, -2)$. This means that $\mathbf{F} \cdot \mathbf{N} = (s + t)^2 - (s - t)^2 - 2st = 2st$. Therefore,

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D 2st\, ds\, dt = \int_0^1 \int_0^{\sqrt{1-t^2}} 2st\, ds\, dt$$

$$= \int_0^1 (s^2 t)\Big|_0^{\sqrt{1-t^2}} dt = \int_0^1 (t - t^3)\, dt = \left(\frac{t^2}{2} - \frac{t^4}{4}\right)\Bigg|_0^1 = \frac{1}{4}.$$

4. (a) You can easily verify that both $\mathbf{X}$ and $\mathbf{Y}$ parametrize the surface $z = 3x^2 + 3y^2$ for $0 \le x^2 + y^2 \le 4$. The major difference is that $\mathbf{X}$ covers the surface once while $\mathbf{Y}$ covers the surface twice.

(b) For $\mathbf{X}$, the standard normal $\mathbf{N}$ is

$$(\cos t, \sin t, 6s) \times (-s \sin t, s \cos t, 0) = (-6s^2 \cos t, -6s^2 \sin t, s)$$

so

$$\iint_{\mathbf{X}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 (s \sin t, -s \cos t, 9s^4) \cdot (-6s^2 \cos t, -6s^2 \sin t, s)\, ds\, dt$$

$$= \int_0^{2\pi} \int_0^2 9s^5\, ds\, dt = \int_0^{2\pi} \frac{9s^6}{6}\Bigg|_0^2 dt = \int_0^{2\pi} 96\, dt = 192\pi.$$

For $\mathbf{Y}$, the standard normal $\mathbf{N}$ is

$$(2 \cos t, 2 \sin t, 24s) \times (-2s \sin t, 2s \cos t, 0) = (-48s^2 \cos t, -48s^2 \sin t, 4s)$$

so

$$\iint_{\mathbf{Y}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}) \cdot d\mathbf{S} = \int_0^{4\pi} \int_0^1 (2s \sin t, -2s \cos t, 144s^4) \cdot (-48s^2 \cos t, -48s^2 \sin t, 4s)\, ds\, dt$$

$$= \int_0^{4\pi} \int_0^1 576s^5\, ds\, dt = \int_0^{4\pi} \frac{576s^6}{6}\Bigg|_0^1 dt = \int_0^{4\pi} 96\, dt = 384\pi.$$

As noted in part (a), the integral over $\mathbf{Y}$ should be twice the integral over $\mathbf{X}$ since they both parametrize the same space but $\mathbf{Y}$ covers the space twice.

8. (a) The sphere is symmetric about the plane $x = 0$. Hence $\iint_S x\, dS = 0$ as for each small piece of the sphere with coordinate $x > 0$ (and $x \le a$), there is a corresponding piece with coordinate $x < 0$. Hence contributions in an appropriate Riemann sum will cancel.

(b) For $x^2 + y^2 + z^2 = a^2$ the outward unit normal is given by $\mathbf{n} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \frac{1}{a}(x + y + z)\, dS$$

$$= \frac{1}{a}\left(\iint_S x\, dS + \iint_S y\, dS + \iint_S z\, dS\right) = 0$$

since each surface integral is zero via reasoning as in part (a).

In Exercises 10–13 we use Definition 2.1: $\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{N}(s,t)\|\, ds\, dt$. *And we'll break down the integral as* $\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$.

10. $\iint_S z\, dS = \int_0^{2\pi} \int_0^4 3t\, dt\, ds + \int_0^{2\pi} \int_0^3 0\, dt\, ds + \int_0^{2\pi} \int_0^3 4t\, dt\, ds = 48\pi + 36\pi = 84\pi.$

14.

$$\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{S} = \int_0^4 \int_0^{2\pi} (3\cos s, 3\sin s, 0) \cdot (3\cos s, 3\sin s, 0)\, ds\, dt$$
$$+ \int_0^3 \int_0^{2\pi} (t\cos s, t\sin s, 0) \cdot (0, 0, -t)\, ds\, dt + \int_0^3 \int_0^{2\pi} (t\cos s, t\sin s, 0) \cdot (0, 0, t)\, ds\, dt$$
$$= \int_0^4 \int_0^{2\pi} 9\, ds\, dt = 72\pi.$$

A different approach would be to observe that as the unit normals for S_2 and S_3 are $\pm\mathbf{k}$ then $\mathbf{F} \cdot \mathbf{n} = 0$ on S_2 and S_3. On S_1 the unit normal is $(x\mathbf{i} + y\mathbf{j})/3$ So $\mathbf{F} \cdot \mathbf{n} = (x^2 + y^2)/3 = 9/3 = 3$. Therefore we obtain, $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = 3(\text{area of } S_1) = 3(2\pi(3)(4)) = 72\pi$.

15. ***Note.*** The example in the lectures was slightly different from the problem in the text; in the lectures, the equation for the lateral surface was $x^2 + y^2 = 4$, and as a result the final answer below is different from the answer for the problem in the lectures.

$$\iint_S (z\mathbf{k}) \cdot d\mathbf{S} = \int_0^4 \int_0^{2\pi} (0,0,t) \cdot (3\cos s, 3\sin s, 0)\, ds\, dt + \int_0^3 \int_0^{2\pi} (0,0,0) \cdot (0,0,-t)\, ds\, dt$$
$$+ \int_0^3 \int_0^{2\pi} (0,0,4) \cdot (0,0,t)\, ds\, dt = \int_0^3 \int_0^{2\pi} 4t\, ds\, dt = \int_0^3 8\pi t\, dt = 36\pi.$$

A different approach would have been to notice that, since the unit normal vector to the lateral surface S_1 has no $\mathbf{k}$ component, $\iint_{S_1} z\mathbf{k} \cdot d\mathbf{S} = 0$. Also, $z = 0$ on S_2 so $\iint_{S_2} z\mathbf{k} \cdot d\mathbf{S} = 0$. Finally, $z = 4$ on S_3 and therefore

$$\iint_S z\mathbf{k} \cdot d\mathbf{S} = \iint_{S_3} z\mathbf{k} \cdot d\mathbf{S} = \iint_{S_3} 4\mathbf{k} \cdot \mathbf{k}\, dS = \iint_{S_3} 4\, dS = 4 \cdot (\text{area of } S_3) = 4(\pi 3^2) = 36\pi.$$

18.

$$\iint_S (x^2\mathbf{i}) \cdot d\mathbf{S} = \int_0^4 \int_0^{2\pi} (9\cos^2 s, 0, 0) \cdot (3\cos s, 3\sin s, 0)\, ds\, dt$$
$$+ \int_0^3 \int_0^{2\pi} (t^2\cos^2 s, 0, 0) \cdot (0,0,-t)\, ds\, dt + \int_0^3 \int_0^{2\pi} (t^2\cos^2 s, 0, 0) \cdot (0,0,t)\, ds\, dt$$
$$= 27 \int_0^4 \int_0^{2\pi} \cos^3 s\, ds\, dt = 27 \int_0^4 \int_0^{2\pi} (1 - \sin^2 s)\cos s\, ds\, dt = 27 \int_0^4 [\sin s - (\sin^3 s)/3]\Big|_0^{2\pi} dt = 0.$$

Section 7.3, p. 453 and following

2. S is a helicoid. We begin by calculating

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We calculated a normal vector in Exercise 20 of Section 7.1: $\mathbf{N} = (\sin t, -\cos t, s)$. So,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\sin t\mathbf{i} - \cos t\mathbf{j} + s\mathbf{k})\, dt\, ds$$
$$= \int_0^1 \int_0^{\pi/2} (\sin t - \cos t + s)\, dt\, ds = \int_0^1 \frac{\pi}{2} s\, ds$$
$$= \frac{\pi}{4} s^2 \Big|_0^1 = \frac{\pi}{4}.$$

On the other hand, ∂S consists of four pieces which we parametrize by $\mathbf{x}_1(s) = (s, 0, 0)$ for $0 \le s \le 1$, $\mathbf{x}_2(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi/2$, $\mathbf{x}_3(s) = (0, 1 - s, \pi/2)$ for $0 \le s \le 1$, and $\mathbf{x}_4(t) = (0, 0, \pi/2 - t)$ for $0 \le t \le \pi/2$. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (0, s, 0) \cdot (1, 0, 0)\, ds + \int_0^{\pi/2} (t, \cos t, \sin t) \cdot (-\sin t, \cos t, 1)\, dt$$
$$+ \int_0^1 (\pi/2, 0, 1 - s) \cdot (0, -1, 0)\, ds + \int_0^{\pi/2} (\pi/2 - t, 0, 0) \cdot (0, 0, -1)\, dt$$
$$= \int_0^1 0\, ds + \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t)\, dt + \int_0^1 0\, ds + \int_0^{\pi/2} 0\, dt$$
$$= \frac{\pi}{4}.$$

These two answers agree.

8. Note that $\nabla \cdot \mathbf{F} = 2x + 2$ so

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 2 \iiint_D (x + 1)\, dV$$
$$= 2 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2+1}^{5} (x + 1)\, dz\, dy\, dx$$
$$= 2 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(x + 1)(4 - x^2 - y^2)]\, dy\, dx$$
$$= \frac{8}{3} \int_{-2}^{2} [(x + 1)(4 - x^2)^{3/2}]\, dx = \frac{8}{3}(6\pi) = 16\pi.$$

On the other hand, the boundary of D can be split into two pieces: the flat top piece S_1 and the surface of the paraboloid S_2. A parametrization of S_1 is $\mathbf{X}_1(s, t) = (t \cos s, t \sin s, 5)$ for $0 \le s \le 2\pi$ and $0 \le t \le 2$. Then a normal vector is $\mathbf{N}_1(s, t) = (0, 0, t)$. A parametrization of S_2 is $\mathbf{X}_2(s, t) = (t \cos s, t \sin s, t^2 + 1)$ for $0 \le s \le 2\pi$ and $0 \le t \le 2$. Then a normal vector is $\mathbf{N}_2(s, t) = (2t^2 \cos s, 2t^2 \sin s, -t)$. So,

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$
$$= \int_0^{2\pi} \int_0^2 (t^2 \cos^2 s, t \sin s, 5) \cdot (0, 0, t)\, dt\, ds$$
$$+ \int_0^{2\pi} \int_0^2 (t^2 \cos^2 s, t \sin s, t^2 + 1) \cdot (2t^2 \cos s, 2t^2 \sin s, -t)\, dt\, ds$$
$$= \int_0^{2\pi} \int_0^2 (2t^4 \cos^3 s + 2t^3 \sin^2 s - t^3 + 4t) = \int_0^{2\pi} \left[8 + \frac{64 \cos^3 s}{5} - 4 \cos 2s\right] ds = 16\pi.$$

These two answers agree.

14. ***<u>The following hint appeared on the assignment sheet:</u>*** Reduce the problem to finding the corresponding surface integral over the disk $\boldsymbol{D}$ defined by $z = \mathbf{1}$ and $\mathbf{0} \le x^2 + y^2 \le \mathbf{1}$, and evaluate the surface integral over $\boldsymbol{D}$.

The key point in the reduction is that the union of D and the original surface S bounds a region V in space such that the divergence of **F** is zero throughout V. By the Divergence Theorem, this implies that the difference of the surface integrals **Int(*S*) – Int(*D*)** must be zero, where both S and D have the upward orientation (note that the orientation which D inherits as the outward normal to V is the ***downward*** orientation, and this is why a negative sign arises). Therefore we know that **Int(*S*) = Int(*D*),** and consequently it is enough to compute **Int(*D*).**

14. S is the portion of the "bell" surface for which $z = e^{1-x^2-y^2}$ and $z \geq 1$. Take S_2 to be the disk in the plane $z = 1$ bounded by the circle $x^2 + y^2 = 1$. Then $S \bigcup S_2$ is the boundary of a solid V. S is oriented with an upward pointing normal and S_2 is oriented with a downward pointing normal.

$$\nabla \cdot \mathbf{F} = 0 \quad \text{so} \quad \iiint_V \nabla \cdot \mathbf{F}\, dV = 0.$$

Also,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} (x, y, 2 - 2z) \cdot (0, 0, -1)\, dS = \iint_{S_2} (2z - 2)\, dS.$$

But along $S_2, z = 1$, so $\iint_{S_2} (2z - 2)\, dS = \iint_{S_2} (2 - 2)\, dS = 0$. So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 - 0 = 0.$$

The next exercise uses the Differentiation Principle stated in the comments to Section 7.4 (see the file **http://math.ucr.edu/~res/math10B/comments0704.pdf**). Specifically, if we are given two functions p and q defined on a region W such that the integrals of p and q over every "nice" closed region in W are equal, then $p = q$.

20. By Gauss's Theorem, $\iiint_D \nabla \cdot (\nabla f) dV = \oiint_{\partial D} (\nabla f) \cdot d\mathbf{S}$. Here the boundary of D consists of finitely many piecewise smooth, closed orientable surfaces S_i. By assumption, $\oiint_{\partial S_i} (\nabla f) \cdot d\mathbf{S} = 0$ and so $\iiint_D \nabla \cdot (\nabla f) dV = 0$. This is true for any solid D, so $\nabla \cdot (\nabla f) = 0$. As we saw earlier in the text $\nabla \cdot (\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$. So f is harmonic.

Section 7.4, p. 467 and following

Note. The result below plays an important role in the derivation of wave motion equations in the theory of electromagnetism (see part **(b)** of this problem on page 468).

15. (a) This is just a straightforward calculation. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = (P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k}$$

and

$$\nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_y - N_z & M_z - P_x & N_x - M_y \end{vmatrix}$$

$$= (N_{xy} - M_{yy} - M_{zz} + P_{xz})\mathbf{i} + (P_{yz} - N_{zz} - N_{xx} + M_{yx})\mathbf{j} + (M_{zx} - P_{xx} - P_{yy} + N_{zy})\mathbf{k}.$$

On the other hand,

$$\nabla(\nabla \cdot \mathbf{F}) = \nabla(M_x + N_y + P_z)$$

$$= (M_{xx} + N_{yx} + P_{zx})\mathbf{i} + (M_{xy} + N_{yy} + P_{zy})\mathbf{j} + (M_{xz} + N_{yz} + P_{zz})\mathbf{k}$$

and $(\nabla \cdot \nabla)\mathbf{F} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{F}$

$$= (M_{xx} + M_{yy} + M_{zz})\mathbf{i} + (N_{xx} + N_{yy} + N_{zz})\mathbf{j} + (P_{xx} + P_{yy} + P_{zz})\mathbf{k}.$$

Hence, $\nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} = (N_{yx} + P_{zx} - M_{yy} - M_{zz})\mathbf{i} + (M_{xy} + P_{zy} - N_{xx} - N_{zz})\mathbf{j} + (M_{xz} + N_{yz} - P_{xx} - P_{yy})\mathbf{k}.$

By assumption $\mathbf{F}$ is of class C^2 and so the mixed partials are equal; thus we have the result:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

Section 7.5, p. 469 and following

2. **False.** All points of the form $\mathbf{X}(s, t)$ lie on the given plane, but if a point can be written as $\mathbf{X}(s, t)$ for some s and t, then its y – coordinate must be at least $\mathbf{3}$ because the latter is equal to $s^2 + 3$. Since the point $(-1, 0, -7)$ lies on this plane, it follows that the plane has at least one point (and, in fact, infinitely many points) which does not have the form $\mathbf{X}(s, t)$ for some values of s and t.

4. **False.** The standard normal is zero when $s = 0$ or $t = 0$.

8. **True.** The integrand is an odd function of $x, y,$ and z — in other words, we have the identity $f(-x, -y, -z) = -f(x, y, z)$ — and furthermore the region is symmetric with respect to the antipodal map sending a vector $\mathbf{v}$ to its negative $-\mathbf{v}$. Under these circumstances we know that the integral of f over the given region must be equal to zero.

22. **False.** By Stokes' Theorem, the surface integral is equal to the line integral of the vector field $\mathbf{F}$ over the boundary curve. Therefore the surface integral of the curl of $\mathbf{F}$ is equal to the line integral of the length $|\mathbf{F}|$ over the boundary curve. The only way this integral can be

zero is if $|\mathbf{F}| = \mathbf{0}$ everywhere, or equivalently if and only if $\mathbf{F} = \mathbf{0}$. Clearly there are many situations in which this condition is not met.

24. **True.** By the Divergence Theorem, the triple integral is equal to the flux (surface) integral of $\mathbf{F}$ over the boundary surface. If this vector field is tangent to the boundary everywhere, then its dot product with the normal vector is always zero, and therefore the integrand of the surface integral is equal to zero, so that the surface integral itself must also be zero.