# III – Matrices and their uses

Linear algebra plays important roles in both theoretical and applied mathematics. Applications generally require some effective computational tools, and the latter usually involve rectangular arrays of scalars known as <u>matrices</u>. Formally, if $m$ and $n$ are positive integers, then an $m \times n$ matrix is a rectangular array with $m$ rows and $n$ columns.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$a_{ij}$ is the entry in row $i$ and column $j$.

To illustrate this, consider the following type of question:

Suppose we are given $n$ specific vectors $v_1, \dots, v_n$ in $\mathbb{R}^n$. Does the set $\{v_1, \dots, v_n\}$ give a basis?

The first step in considering this problem is to construct an $n \times n$ matrix whose $i^{th}$ row is $v_i$.

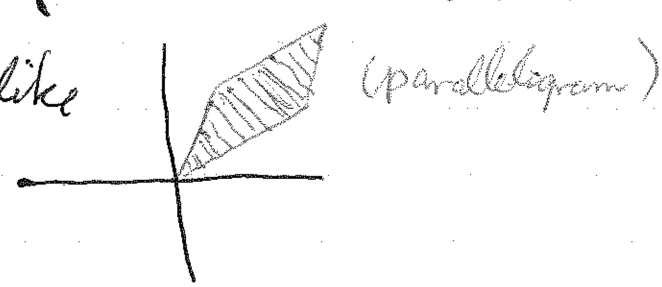
In this and other problems, conceptual issues ~~are~~ require more attention than in calculus or differential equations, and this is one reason why theory plays such a large role in most linear algebra courses.

Since the book by Axler is definitely tilted towards the theory of linear algebra, one goal of the lectures will be to put a little more emphasis on $\boxed{\text{some}}$ of the basic computational techniques that can be used to study applications.

## Linear transformations

If V and W are vector spaces over a field F, a <u>linear transformation</u> T from V to W (written $T: V \to W$) is a function/mapping from V to W such that vector addition & scalar multiplication are preserved.

In multivariable calculus, such maps from $\mathbb{R}^n$ to itself are used to convert an integral over a region like




to an integral over a rectangular region via the change of variables formula.

There are many examples in Axler. Here is another. Consider the plane rotation map on $\mathbb{R}^2$ (to itself) sending a point with the polar coords $(r, \theta)$ to $(r, \theta+\alpha)$ for some fixed $\alpha$. Standard trigonometric identities imply that a point with rectangular coordinates $(x, y)$ is sent to $(\cos\alpha \cdot x - \sin\alpha \cdot y,\ \sin\alpha \cdot x + \cos\alpha \cdot y)$,

which is a special case of an example in Axler.

Here are the messy details. Write $(x, y) = (r\cos\theta, r\sin\theta)$ using polar coordinates. Then we have

$(r\cos(\theta+\alpha), r\sin(\theta+\alpha)) =$

$(r(\cos\theta\cos\alpha - \sin\theta\sin\alpha),\ r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)) =$

$((r\cos\theta)\cos\alpha - (r\sin\theta)\sin\alpha,\ (r\sin\theta)\cos\alpha + (r\cos\theta)\sin\alpha) =$

$(x\cos\alpha - y\sin\alpha,\ y\cos\alpha + x\sin\alpha)$, which is the expression given above.

Definition. $\mathcal{L}(V, W)$ = linear transformations from $V$ to $W$. It has natural notions of addition & scalar multiplication (pointwise).

A NONEXAMPLE. Let $V = W = \mathbb{R}$, $T(x) = x^3$.

Then $T(cx) = c^3 T(x)$, not $cT(x)$, and $T(x+y) \neq T(x) + T(y)$ in most cases because $(x+y)^3$ is usually not equal to $x^3 + y^3$.

Composition (products) of linear transformations

Given $T_1 : V \to W$ and $T_2 : W \to X$ linear, we have $T_2 \circ T_1 : V \to X$ with $T_2(T_1 v) = T_2 \circ T_1 (v)$. — This operation and the previous ones ~~stan~~ satisfy standard identities like $T_3 \circ (T_2 \circ T_1) = (T_3 \circ T_2) \circ T_1$, distributivity laws, $Id_W \circ T = T \circ Id_V = T$. However, in general $T_1 \circ T_2 \neq T_2 \circ T_1$. In fact, this is not even true for some examples such that

$$\text{source}(T_1) = \text{target}(T_2), \ \text{source}(T_2) = \text{target}(T_1)$$

so that both products are defined.

<u>Kernels + images</u> $T: V \to W$ linear

Kernel $(T)$ = all $v \in V$ such that $Tv = 0$.

Image $(T)$ = all $w \in W$ such that $w = Tv$, some $v \in V$.

These are subspaces of $V$ + $W$ respectively.

<u>Prop.</u> If $T$ is 1-1 and onto, then $T^{-1}$ is also linear.

<u>Special case where $\dim V < \infty$.</u>

<u>Prop.</u> Let $v_1, \dots, v_n$ be a basis for $V$, and let $w_1, \dots, w_n \in W$. Then there is a unique linear transformation $T: V \to W$ such that $Tv_i = w_i$ for all $i$.

<u>Thm.</u> Suppose $T: V \to W$ where $V$ and $W$ are finite dimensional. If rank $T = \dim \operatorname{Im} T$ and nullity $T = \dim \operatorname{Ker} T$, then

$$\dim V = \text{rank } T + \text{nullity } T.$$

<u>Corollary</u> If $\dim V = n$, then there is an invertible linear transformation $T: F^n \to V$.

<u>Proof.</u> Let $v_1, \ldots, v_n$ be a basis for $V$, let $e_1, \ldots, e_n$ be the standard basis for $F^n$, and consider the linear transformation $T$ sending $e_i$ to $v_i$ for all $i$. Then $T$ is onto, for

$x = \sum c_i v_i \Rightarrow x = T(c_1, \ldots, c_n)$. By the theorem, nullity $= 0$. <u>CLAIM</u> $T$ is 1-1: If $Tx = Ty$, then $T(x-y) = Tx - Ty = 0$, so $x - y \in \operatorname{Ker} T = \{0\}$, which means $x = y$. Hence $T$ is 1-1 onto, and by the first prop. $T$ has a linear inverse.

<u>Thm.</u> There is a 1-1 correspondence

$$\mathcal{L}(F^n, F^m) \longleftrightarrow m \times n \text{ matrice such that}$$

$$\text{addition} \longleftrightarrow \text{entrywise addition}$$

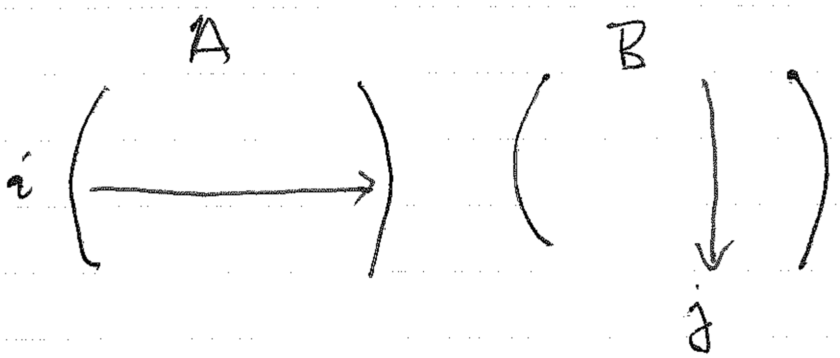
$$\text{scalar multiplication} \longleftrightarrow \text{entrywise scalar multiplication}$$

## Computations with matrices

Let $A$ & $B$ be $n \times m$ and $p \times n$ matrices$/F$, and let $T_A : F^m \longrightarrow F^n$, $T_B : F^n \longrightarrow F^p$ be the corresponding linear transformations. Then

$T_B \circ T_A = T_C$, where $C$ is the $m \times p$ matrix

with entries $$c_{i,j} = \sum_{k=1}^{n} b_{i,k}\, a_{k,j}$$




Rule of index fingers multiplication.

## Row operations on matrices.

1. Multiply a ~~matrix~~ row by a nonzero scalar
2. Interchange two rows.
3. Add a multiple of one row to another.

Two matrices are <u>row-equivalent</u> if one can be obtained from the other by a finite sequence of elementary row operations.

<u>Thm 1.</u> If $A$ and $B$ are row equivalent, then
$\operatorname{Ker} T_A = \operatorname{Ker} T_B$
(It is enough to do this when $B$ is obtained from $A$ by a single operation. Use finite induction to derive the cases involving multiple operations)

<u>Thm. 2</u> Every $m \times n$ matrix $A$ is row-eq to a unique row reduced echelon matrix $C$ s.t.

① For each nonzero row $i$, the first nonzero entry equals $1$

② If row $i$ is nonzero and $p(i)$ is the column containing the first nonzero entry, then every other entry in column $p(i)$ is zero.

③ If $r$ is the number of nonzero rows, then the first $r$ rows are nonzero (+ the others are zero).

④ $p(1) < p(2) < \dots < p(r)$.

Thm. 3 If A and B are row-equivalent and Row(A), Row(B) are the subspaces spanned by their rows, then Row(A) = Row(B) [again, it suffices to consider the case where B is obtained by a single row operation].

Thm 4 Two matrices are row-equivalent $\iff$ their row spaces are equal.

The dimension of the row space is called the row rank.

Solving the system $AX = 0$

$$\left(\sum a_{ij} x_j = 0 \quad i = 1, \ldots, m\right)$$

Put A in row reduced echelon form (by, say, Gaussian elimination). If E is this form, then there are $n-r$ linearly indep solutions; the coords $p(1), \ldots, p(r)$ are determined by the others, and the latter may vary arbitrarily.

Inhomogeneous systems $AX = B \leftarrow$ $n \times 1$ column vector

Put $(\overset{n}{A} \vdots \overset{1}{B})$ in r.r.e. form

① If $n+1 \neq p(i)$ for every $i$, then one can read off the solution as in the homogeneous case.

② If $n+1 = p(i)$ for some $i$, then the system has no solutions; the attempt to solve it yields the false conclusion $0=1$.

<u>Example</u> Consider the system $\begin{matrix} x-y=0 \\ x-y=1 \end{matrix}$, so

$$A = \left(\begin{array}{cc:c} 1 & -1 & 0 \\ 1 & -1 & 1 \end{array}\right) \text{ which is row equiv to } \left(\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

<u>Invertibility</u>

<u>Cor.</u> If $A$ is $n \times n$ <u>and</u> in r.r.e. form <u>and</u> $A$ is invertible, then $A = I$.

Leads to...

<u>Invertibility test.</u> Is $A$ invertible? If so, find $A^{-1}$.

<u>Method</u> Put $(\overset{n}{A} \vdots \overset{n}{I})$ into rre form

$(\underset{n}{E} \vdots \underset{n}{B})$.

① If $E \neq I$, then $A$ is not invertible.

② If $E = I$, then $A$ is invertible and $B = A^{-1}$.

<u>Prop.</u> (should have been earlier) ① Let $A$ be $n \times n$. Then $A$ is invertible $\Leftrightarrow$ $T_A$ is, in which case $(T_A)^{-1} = T_{A^{-1}}$.

② If $m < n$, then $T_A$ is not 1-1.

③ If $n < m$, then $T_A$ is not onto.

<u>Column operations</u> Analogous to row operations. In particular, we can talk about the column rank of a matrix.

<u>Thm.</u> row rank $A$ = column rank $A$.

Note that the column space of $A$ is the image of $T_A$.

See Axler for other useful computational facts.

TRANSPOSITION The final two subsections of Chapter 3 in Axler will not be needed in this course, but one related notion will be.

If $A$ is an $m \times n$ matrix, its transpose ${}^{T}A$ is the $n \times m$ matrix such that $({}^{T}A)_{i,j} = a_{j,i}$

So the rows of $A$ are the columns of ${}^{T}A$ and vice versa.

Properties of transposition

1. ${}^{T}({}^{T}A) = A$
2. ${}^{T}(A+B) = {}^{T}A + {}^{T}B$
3. ${}^{T}(cA) = c \cdot {}^{T}A$
4. ${}^{T}I = I$
5. ${}^{T}(AB) = {}^{T}B \cdot {}^{T}A$ Note the reversal of order!

Chapter 4 of Axler summarizes basic facts about polynomials. These will be ~~stu~~ discussed when they are needed throughout the course.