5A - Eigen$\left\{\begin{matrix}\text{vectors}\\ \text{values}\end{matrix}\right\}$ and invariant subspaces

The main goal of Ch. 5 is to understand linear transformations better. In particular, we want to show that one can often change bases for a finite dimensional vector space V so that a linear transformation $T: V \to V$ looks simpler in the new basis. The focal point will be linear transformations for which one can find a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ $(n = \dim V)$ such that $Tv_i = cv_i$ for some scalars c_i.

We shall say that T is <u>diagonalizable</u> if such a basis exists, for if we define the $n \times n$ matrix of some $S: V \to V$ by

$$S v_j = \sum a_{ij} v_i,$$

then the matrix of $S = T$ as above will only have nonzero entries down the main diagonal:

$$\begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix}.$$

Not every $T: V \to V$ is diagonalizable, but "most" turn out to be so in a sense we shall describe later.

Here are some formalities.

<u>Def.</u> Let $T: V \to V$ (possibly $\dim V = \infty$). A nonzero $v \in V$ is said to be an <u>eigenvector</u> for T if $Tv = cv$ for some scalar c. The number c is called an <u>eigen value</u> for T.

Some writers who prefer not to use an amalgamation of German and English syllables write characteristic $\left\{\begin{matrix}\text{value}\\ \text{vector}\end{matrix}\right\}$ or proper $\left\{\begin{matrix}-\\ -\end{matrix}\right\}$ instead.

If c is an eigenvalue, then the eigenspace $V_{c,T}$ or V_c is $\{0\}$ together with all eigenvectors for c (with respect to T, of course).

<u>General facts</u> (i) V_c is a (vector) subspace. (ii) T maps V_c into itself (i.e., V_c is a T-<u>invariant subspace</u>).

<u>VERIFICATIONS.</u> First, $T0 = 0 = c0$, so V_c = all $v \in V$ such that $Tv = cv$.

(i) $Tv = cv$ & $Tw = cw \Rightarrow T(v+w) = Tv + Tw = cv + cw = c(v+w)$, and $Tv = cv$, $a \in F \Rightarrow Tav = aTv = a(cv) = c(av)$. $\blacksquare$

(ii) Suppose $v \in V_c$, so that $Tv = cv$.

~~Then T sends av to aTv = acv = c(av), so~~

Then $Tv = cv$ also lies in V_c because $T(Tv) = T(cv) = cTv$. $\blacksquare$

<u>Proposition</u> Let $T: V \to V$. Then c is an eigenvalue for $T \Longleftrightarrow$ Kernel $(T - cI)$ is nonzero.

<u>Proof.</u> ($\Leftrightarrow$) $Tv = cv$ with $v \neq 0 \Leftrightarrow (T - cI)v = 0 \Leftrightarrow 0 \neq v \in$ Kernel $T - cI$. $\blacksquare$

<u>Cor.</u> If V is fin dim, c is an eigenvalue $\Leftrightarrow$ $(T - cI)$ is not 1-1, and also $\Leftrightarrow$ $(T - cI)$ is not onto.

Finding eigenvalues + eigenvectors.

Example in Axler proves rigorously that T_A has no $\boxed{\text{real}}$ eigenvalues if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, but it has $\boxed{\text{complex}}$ eigenvalues.

Here are examples of matrices with bases of real eigenvectors.

1. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. To find eigenvalues we must find scalars c such that $\begin{pmatrix} 1-c & 2 \\ 2 & 1-c \end{pmatrix}$ is not invertible. Do this by trying to put the matrix into row-reduced echelon form. Alternatively, in the 2×2 case it suffices to show that one row is a scalar multiple of the other: $(1-c, 2) = k(2, 1-c)$ or $\frac{1-c}{2} = k = \frac{2}{1-c}$. Solving this proportionality equation, we get $(1-c)(1-c) = 4$, or

$c^2 - 2c - 3 = 0$. This factors as $(c+1)(c-3)$ so eigenvalues exist and are $3, -1$.
We can now check that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvalue for $+3$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvalue for -1.

2. $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$, so $A - cI = \begin{pmatrix} -c & -2 \\ 1 & 3-c \end{pmatrix}$.

In this case the proportionality condition becomes $c^2 - 3c = -2$ or $c^2 - 3c + 2 = 0$ or $(c-2)(c-1) = 0$. The eigenvectors for $+2$ are the nonzero solutions for $\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ [nonzero multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$] and the eigenvectors for $+1$ are the nonzero ~~multiples~~ solutions of $\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ [multiples of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$].

<u>General rule for 2×2 case.</u> If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

the eigenvalues are those r such that $(a-r, b)$ and $(c, d-r)$ are lin dep., or equivalently the determinant

$$\boxed{\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} \text{ is zero.}}$$

(We assume 2×2 and 3×3 determinants are known from "College Algebra")

Final remarks

Theorem If $v_1, \dots, v_k$ are eigenvectors for distinct eigenvalues $c_1, \dots, c_k$ then $\{v_1, \dots, v_k\}$ is linearly independent.

Proof $k=1$, no problem. Assume true for $(k-1)$ vectors + proceed by induction. If the set is not linearly independent, then, say, v_k is a linear combination of the previous vectors, so that

$v_k = \sum_{j<k} a_j v_j$. Then

$c_k v_k = \sum c_k a_j v_j$ on one hand, but also

$c_k v_k = T v_k = \sum a_j T v_j = \sum c_j a_j v_j$.

Since $v_k \neq 0$, some $a_{j_0} \neq 0$. By linear independence, $c_k a_{j_0} = c_{j_0} a_{j_0}$, so that $c_k = c_{j_0}$, which contradicts the assumption that $c_1, \ldots, c_k$ were distinct. Hence $\{v_1, \ldots, v_k\}$ is linearly independent, completing the inductive step. ∎

<u>Corollary</u> If $\dim V = n$, then T has at most n distinct eigenvalues. ∎

If there are in fact n distinct eigenvalues, then T must be diagonalizable. ∎

A nondiagonalizable example

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad a = \text{any scalar } a \neq 0$$

Look at $|A - rI|$ = determinant $\begin{vmatrix} 1-c & a \\ 0 & 1-c \end{vmatrix}$

$= (1-c)^2$. So $c = 1$ is the only eigenvalue.

Then $(A - I) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \Rightarrow$ Kernel $A - I$

spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So there is no basis of

eigenvectors in this example, either over

$\mathbb{R}$ or $\mathbb{C}$.