# 5B. Upper triangular form

Recall Axler's example $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A$.

Then $|A - cI| = \begin{vmatrix} -c & -1 \\ 1 & -c \end{vmatrix} = c^2 + 1$

which means that A has no real eigenvalue, but it does have two complex eigenvalues $\pm i$ where $i^2 = -1$.

<u>Theorem</u> If A is an $n \times n$ matrix over $\mathbb{C}$, then A has at least one complex eigenvalue.

<u>Proof</u> Based on

FUNDAMENTAL THEOREM $\left\{\begin{matrix}\text{OF} \\ \text{ABOUT}\end{matrix}\right\}$ ALGEBRA.

Every complex polynomial of degree $n > 0$ factors as $K(x-c_1)\ldots(x-c_n)$ for some complex numbers $c_j$ and $K \neq 0$.

It's really <u>about</u> algebra because all proofs use geometric/topological properties of the complex plane.

$\boxed{\dim V = n}$

<u>On to the proof.</u> Let $v \neq 0$. Then the sequence of vectors $v, Tv, \ldots, T^n v$ is linearly dependent. Choose $m$ to be the first positive integer such that $\{v, \ldots, T^m v\}$ is lin. dep.; since $\{v\}$ is linearly indep, such an integer exists. Then we have $T^m v + \sum_{j=0}^{m-1} c_j T^j v = 0$

Factor $p(t) = t^m + \sum c_j t^j$ as in the Fund. Thm. of Algebra. $p(t) = K \cdot \prod (t - c_j)$

product

If $v_0 = v$ and $v_j = (T - c_j I) v_{j-1}$, then $v_0 \neq 0$ but $v_m = 0$. Let $k$ be the first $j > 0$ such that $v_k = 0$ (by the formula, all subsequent vectors are also zero). Then

$0 = v_k = (T - c_{k-1} I) v_{k-1} \implies T v_{k-1} = c_{k-1} v_{k-1}$.

By choice, $v_{k-1} \neq 0$ & hence is an eigenvector associated to $c_{k-1}$. ∎

For Axler's example, the eigenvectors for $\pm i$ are given by $\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$. ∎

This is a special case of

<u>Proposition</u> Let $A$ be an $n \times n$ matrix with real entries, and suppose that $Z = \overset{\text{real}}{X} + \overset{\text{real}}{Y} i$ is a complex eigenvector for the $\overline{\text{complex}}$ eigenvalue $c = a + bi$. Then $\overline{Z}$ is a complex eigenvector for $\overline{c}$.

<u>Proof.</u> If we form matrices $\overline{A}, \overline{B}$ from $A$ & $B$ by conjugating $\overline{\text{entries}}$ and $AB$ is defined, then $\overline{A} \cdot \overline{B} = \overline{AB}$. Therefore

$$AZ = cZ \implies \overline{c}\,\overline{Z} = \overline{AZ} = \overline{A} \cdot \overline{Z},$$

and the RHS is $A\overline{Z}$ because $A$ has real entries. ∎

<u>Triangular matrices</u>

$$\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \qquad \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

upper $\qquad$ lower

Def. An $n \times n$ matrix is $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ triangular if $a_{ij} = 0$ when $\begin{Bmatrix} i > j \\ i < j \end{Bmatrix}$.

Theorem (TRIANGULAR FORM) Let $\dim V = n < \infty$, and let $T: V \to V$ be a linear transformation. Then there is an ordered basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ such that the matrix $A$ defined by

$$T v_j = \sum a_{ij} v_i \quad \text{is}$$

upper triangular. (over the complex numbers!!)

Proof. $\dim V = 1$, no problem with any basis $\{v_1\}$, for $T v_1 = c_1 v_1$ some $c_1$.

Assume by induction that the result is true for $\dim V = n-1$. By the previous theorem, $T$ has an eigenvector $v_1$ such that $T v_1 = c_1 v_1$ for some $v_1$.

Expand $\{v_1\}$ to a basis $\{v_1, w_2, \ldots, w_n\}$.

(n-1)-dimensional

Let $W = \text{span}\{w_2, \ldots, w_n\}$, so that we may write

$$Tw_j = \left(\sum_2^n b_{ij} w_i\right) + b_{1j} v_1 .$$

Define $T_0 : W \to W$ by dropping the last term. By induction we can find a basis for $W$, $\{v_2, \ldots, v_n\}$ such that

$$T_0 w_j = \sum_2^n p_{ij} v_i \quad \text{where } p_{ij} = 0 \text{ if}$$

$i > j$. Then if we take $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ we have $Tv_j = \sum a_{ij} v_i$ where $a_{ij}$ is

$$\begin{array}{ll} p_{ij} & \text{if } i, j \geq 2 \\ b_{1,j} & \text{if } i = 1,\ j = 2 \\ c_1 & \text{if } i = j = 1. \\ 0 & \text{if } i \geq 2,\ j = 1. \end{array}$$

By the definitions we have $a_{ij} = 0$ if $i > j$. $\blacksquare$

<u>Corollary</u> If $T$ is in triangular form and $c$ is an eigenvalue, then $c$ is a diagonal entry of the upper triangular matrix $A$.

<u>Proof.</u> One can show that $(A-cI)$ is invertible $\iff$ $c$ is not a diagonal entry, or equivalently $(A-cI)$ is not invertible $\iff$ $c$ is a diagonal entry. ■

<u>Examples</u> 1. The only eigenvalue of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $1$, and there is only one independent eigenvector, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. $\boxed{\text{WARNING}}$ The unit vectors need not be the the eigenvectors. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ the eigenvalues are $1$ and $2$, and their eigenspaces are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively!

<u>Final Remark</u> (Not in the book, but worth knowing) Every linear transformation $T: V \to V$ (over $\mathbb{C}$, with $\dim V < \infty$) can be approximated by one with a basis of eigenvectors.

<u>Construction</u> May as well restrict to up. triangular matrices. If $A$ is such a matrix, alter the diagonal entries $a_{jj}$ by numbers $h_j$ which are <u>sufficiently small</u>* and such that $\{a_{11}+h_1, \ldots, a_{nn}+h_n\}$ are distinct. ∎

* This can be made precise, but we shall avoid doing so here.

A similar result holds over the reals, but it is more complicated by the lack of suitable triangular forms.