5C. More on diagonalization

Thm. If $T: V \to V$ has $n = \dim V$ $\boxed{\text{distinct}}$ eigenvalues, then T is diagonalizable.

Proof. Let $v_1, \dots, v_n$ be eigenvectors for the distinct eigenvalues $c_1, \dots, c_n$. Then we know $\{v_1, \dots, v_n\}$ is linearly independent, and since $n = \dim V$ this set must form a basis for V. ∎

Axler, Example 5.45

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

By a previous result, the eigenvalues are 2, 5, 8.

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 6 \end{pmatrix} \qquad A - 5I = \begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

eigenvector for this value is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

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$$A - 8I = \begin{pmatrix} -6 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{eigenvector} \begin{pmatrix} 1 \\ +6 \\ +6 \end{pmatrix}.$$

Problem. For what real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can one find a basis of real eigenvectors.

Solution Eigenvalues are the roots of $t^2 - (a+d)t + (ad-bc) = 0$, and this equation has two real roots if the discriminant $B^2 - 4AC$ is positive, so the answer is that there are two real roots $\Leftrightarrow$

$$(a+d)^2 > 4(ad-bc).$$

Over the complex numbers, there are two roots $\Leftrightarrow (a+d)^2 \neq 4(ad-bc)$.

Iterations of linear transformations

There are many situations in which we have a vector $v \in V$, a linear transformation $T: V \to V$, and a reason for considering the iterated powers of T acting on v:

$v, Tv, T^2 v, \ldots$

If we have a basis of eigenvectors for T we can do so as follows.

1. Find [the] eigenvectors $x_1, \ldots, x_n$ forming a basis along with their associated eigenvalues c_i.
2. Find the coefficients a_i such that $v = \sum_i a_i x_i$
3. Conclude that $T^k v = \sum_i a_i T^k x_i = \sum_i a_i c_i^k x_i$.

Previous example continued

$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Have seen $A\begin{pmatrix} 2 \\ -1 \end{pmatrix} \overset{1\cdot}{=} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find $A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Step ① is already completed.

For Step 2, write $\begin{pmatrix}1\\0\end{pmatrix} = p\begin{pmatrix}2\\-1\end{pmatrix} + q\begin{pmatrix}1\\-1\end{pmatrix}$ & solve for p, q. Use any valid method & get $p = 1$, $q = -1$: $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}$.