For Step 2, write $\begin{pmatrix}1\\0\end{pmatrix} = p\begin{pmatrix}2\\-1\end{pmatrix} + q\begin{pmatrix}1\\-1\end{pmatrix}$ & solve for $p, q$. Use any valid method & get $p = 1$, $q = -1$: $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}$.

Then $A^k\begin{pmatrix}1\\0\end{pmatrix} = 2^k\begin{pmatrix}2\\-1\end{pmatrix} - \cancel{1}^k\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2^{k+1}-1\\-2^k-1\end{pmatrix}$

## APPLICATION TO AMORTIZED LOANS

SETTING. A sum $S$ of money is borrowed for $Y$ years at an annual interest rate of $q$ percent. The money is to be repaid in equal monthly installments, with interest charged only on the remaining balance. Find the monthly payments $P$.

Let $N = 12Y$ be the number of payments, and let $r = \frac{q}{1200}$ (monthly interest converted to fraction). Then we have

$$x_{k+1} = x_k - \underset{\substack{\uparrow \\ \text{subtract} \\ \text{payment}}}{P} + \underset{\substack{\uparrow \\ \text{add back} \\ \text{interest charges}}}{r x_k}$$

where $x_k$ = monthly balance at month $k$.

We also know that $x_0 = S$ and $x_N = 0$.

Rewrite everything in matrix form:

$$\underbrace{\begin{pmatrix} x_{k+1} \\ 1 \end{pmatrix} = \begin{pmatrix} 1+r & -P \\ 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_k \\ 1 \end{pmatrix}$$

Then we want to solve for P. This requires finding a formula for $\begin{pmatrix} x_k \\ 1 \end{pmatrix} = A^k \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$.

(1) Find a basis of eigenvectors for A.

The eigenvalues are easy to find because $1+r$ and $1$ are distinct ($r > 0$, of course).

~~They~~ eigenvectors are $V_1 = \begin{pmatrix} P \\ r \end{pmatrix}$ $V_{1+r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

[ Solve $(A - cI)(W) = 0$ for $c = 1, 1+r$ ]

(2) Express $\begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} S \\ 1 \end{pmatrix}$ as a linear combination of these. In fact, we have

$$\begin{pmatrix} x_0 \\ 1 \end{pmatrix} = \frac{1}{r} V_1 + \left[ S - \frac{P}{r} \right] V_{1+r}.$$

This immediately yields

$$\begin{pmatrix} x_k \\ 1 \end{pmatrix} = A^k \begin{pmatrix} S \\ 1 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} P \\ r \end{pmatrix} + (1+r)^k \left[S - \frac{P}{r}\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

but we still need to find P. To do so, use the condition $x_N = 0$, so that

$$\frac{P}{r} + (1+r)^n \left[S - \frac{P}{r}\right] \cancel{1} = 0.$$

Solving this linear eqn. for P, we get

$$P = \frac{r S (1+r)^N}{(1+r)^N - 1}. \blacksquare$$