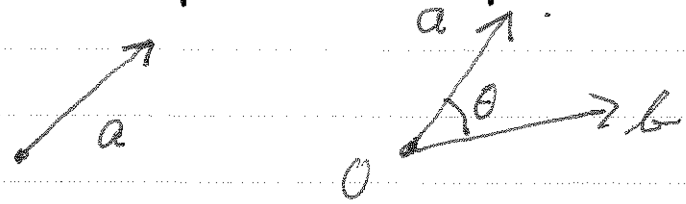
6A. Inner (or Dot) products

If $n = 2$ or 3, the DOT PRODUCT on $\mathbb{R}^n$ is given as follows. Given a, b in $\mathbb{R}^n$, write their coordinates as a_i + b_i respectively. Then $a \cdot b = \underset{\substack{\text{MORE COMMON} \\ \text{IN ADV. MATH.}}}{\langle a, b \rangle} = \sum_{i=1}^{2 \text{ or } 3} a_i b_i$.

We can use dot products to express important geometric information algebraically.



<u>Length</u> $|a| = \sqrt{a \cdot a}$ (Pythagorean Thm.)

$\uparrow$ positive, zero $\Leftrightarrow a = 0$

<u>Angle</u> $\cos\theta = \dfrac{a \cdot b}{|a| \cdot |b|}$ (Law of Cosines)

Dot products satisfy simple identities

$a \cdot b = b \cdot a$, $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$

$(a_1 + a_2) \cdot (b_1 + b_2) = a_1 \cdot b_1 + a_2 \cdot b_1 + a_1 \cdot b_2 + a_2 \cdot b_2$.

$a \cdot a \geq 0$, with equality $\Leftrightarrow a = 0$.

Other identities like $0 \cdot a = 0 = a \cdot 0$ follow from these.

In $\mathbb{R}^3$ the cross product is useful, but it doesn't extend to higher dimensions in a simple fashion.

<u>Inner product in $\mathbb{R}^n$</u> If $a, b \in \mathbb{R}^n$, then $\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$.

It is an "exercise in bookkeeping" to check that $\langle a, b \rangle$ has all the previously listed alg-properties of the dot product. We shall see it also has the <u>geometric</u> properties of the dot product given above.

<u>Length</u> $|a| = \sqrt{a \cdot a}$ (unique nonneg sqrt).

<u>Angle</u> $\theta = \arccos \dfrac{a \cdot b}{|a| \cdot |b|}$.

<u>Immediate question</u>. How do we know the latter makes sense? — In particular, why is the quotient always between -1 and $+1$?

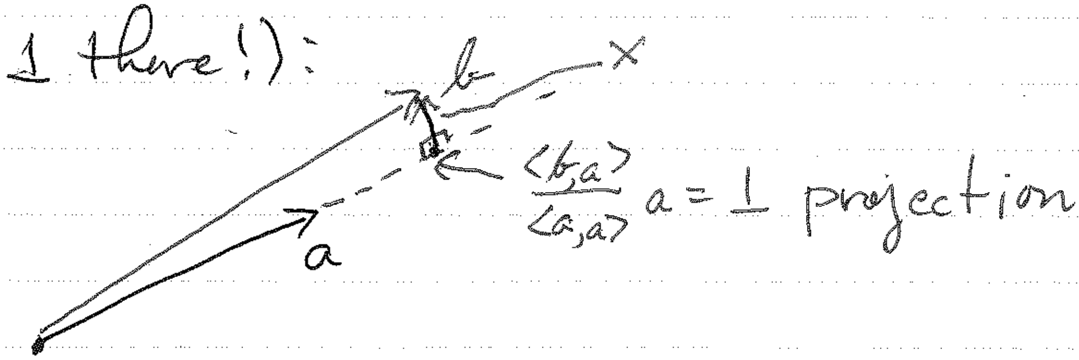
Cauchy–Schwarz–Buniakovski Inequality

$$|a \cdot b| \le |a| \cdot |b| \quad \text{with}$$

equality $\Longleftrightarrow$ a & b are linearly dependent.

The conclusion follows immediately if $a = 0$ or $b = 0$, so assume in the derivation that $a, b \neq 0$.

Derivation. Let $x = b - \frac{\langle b,a\rangle}{\langle a,a\rangle} a$. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the vector x is perpendicular to the line containing 0 and a (also in higher dims, once we define $\perp$ there!):



We know that $\langle x, x\rangle \ge 0$. Now substitute the definition of x to get $\left\langle b - \frac{\langle b,a\rangle}{\langle a,a\rangle} a,\ b - \frac{\langle b,a\rangle}{\langle a,a\rangle} \right\rangle =$
$|b|^2 - 2\frac{\langle b,a\rangle}{\langle a,a\rangle}\langle b,a\rangle + \frac{\langle b,a\rangle^2}{|a|^4}|a|^2 \ge 0.$

We can rewrite this as $|b|^2|a|^2 \ge \langle a, b\rangle^2$, and taking square roots yields the CSB $\leq$. $\blacksquare$

Inner product on $\mathbb{C}^n$ We have to give up something, and experience shows it's